CAREER POINT

JEE Advanced Exam 2023
(Paper & Solution)

Date : 04 / 06 / 2023

PAPER-2

MATHEMATICS

SECTION – 1 (Maximum Mark : 12)

- This section contains **FOUR (04)** questions.
- Each question has four options (A), (B), (C) and (D). **Only one** of these four option is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	: **+3**	If **ONLY** the correct option is chosen;
Zero Marks	: **0**	If none of the option is chosen (i.e. the question is unanswered);
Negative Marks	: –1	In all other cases.

Q.1 Let f : [1, ∞) → R be a differentiable function such that $f(1) = \frac{1}{3}$ and $3\int_1^x f(t)dt = xf(x) - \frac{x^3}{3}, x \in [1,\infty)$.

Let e denote the base of the natural logarithm. Then the value of f(e) is

(A) $\frac{e^2+4}{3}$ (B) $\frac{\log_e 4 + e}{3}$ (C) $\frac{4e^2}{3}$ (D) $\frac{e^2-4}{3}$

Ans. **[C]**

Sol. $3\int_1^x f(t)dt = xf(x) - \frac{x^3}{3}$

$\Rightarrow 3f(x) = f(x) + xf'(x) - x^2$

$\Rightarrow xf'(x) - 2f(x) = x^2$

$\Rightarrow f'(x) - \frac{2}{x}f(x) = x \Rightarrow$ (Linear differential equation)

$\Rightarrow$ I.F. $= e^{-\frac{2}{x}dx} = \frac{1}{x^2}$

$\Rightarrow y\left(\frac{1}{x^2}\right) = \int x \times \frac{1}{x^2}dx = \text{In}x + C$

$\Rightarrow y = x^2(\text{In}x + C)$

$\Rightarrow f(x) = x^2(\text{In}x + C)$

$\Rightarrow f(1) = 1\ (0 + C) \Rightarrow C = \frac{1}{3}$

$\Rightarrow f(e) = e^2\left(\text{In}e + \frac{1}{3}\right)$

$\Rightarrow f(e) = \frac{4e^2}{3}$

Q.2 Consider an experiment of tossing a coin repeatedly until the outcomes of two consecutive tosses are same. If the probability of a random toss resulting in head is $\frac{1}{3}$, then the probability that the experiment stops with head is

(A) $\frac{1}{3}$ (B) $\frac{5}{21}$ (C) $\frac{4}{21}$ (D) $\frac{2}{7}$

Ans. **[B]**

Sol. $P(H) = \frac{1}{3}$

$P(T) = \frac{2}{3}$

P(E) = P(HH) + P(THH) + P(HTHH) + P (THTHH) + P(HTHTHH) + P(THTHTHH) + ...

$= \frac{1}{3^2} + \frac{2}{3^3} + \frac{2}{3^4} + \frac{4}{3^5} + \frac{4}{3^6} + \frac{8}{3^7} + \frac{8}{3^8} + ...$

$= \left(\frac{1}{3^2} + \frac{2}{3^4} + \frac{4}{3^6} + ...\right) + \left(\frac{2}{3^3} + \frac{4}{3^5} + \frac{8}{3^7} + ...\right)$

$P(E) = \frac{1}{7} + \frac{2}{21} = \frac{5}{21}$

Q.3 For any $y \in R$, let $\cot^{-1}(y) \in (0,\pi)$ and $\tan^{-1}(y) \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. Then the sum of all the solutions of the equation $\tan^{-1}\left(\frac{6y}{9-y^2}\right) + \cot^{-1}\left(\frac{9-y^2}{6y}\right) = \frac{2\pi}{3}$ for $0 < |y| < 3$, is equal to

(A) $2\sqrt{3} - 3$ (B) $3 - 2\sqrt{3}$ (C) $4\sqrt{3} - 6$ (D) $6 - 4\sqrt{3}$

Ans. **[C]**

Sol. $\tan^{-1}\left(\frac{6y}{9-y^2}\right) + \cot^{-1}\left(\frac{9-y^2}{6y}\right) = \frac{2\pi}{3}$ $0 < |y| < 3 \Rightarrow y \in (-3, 3) - \{0\}$

Case –I: $\frac{6y}{9-y^2} > 0 \Rightarrow y > 0$

$\tan^{-1}\left(\frac{6y}{9-y^2}\right) + \tan^{-1}\left(\frac{6y}{9-y^2}\right) = \frac{2\pi}{3}$

$\Rightarrow 2\tan^{-1}\left(\frac{6y}{9-y^2}\right) = \frac{2\pi}{3}$

$\Rightarrow \tan^{-1}\left(\frac{6y}{9-y^2}\right) = \frac{\pi}{3} \Rightarrow \frac{6y}{9-y^2} = \sqrt{3}$

$\Rightarrow 6y = 9\sqrt{3} - \sqrt{3}\,y^2$

$\Rightarrow \sqrt{3}\,y^2 + 6y - 9\sqrt{3} = 0$

$\Rightarrow \sqrt{3}\,y^2 + 9y - 3y - 9\sqrt{3} = 0$

$\Rightarrow \sqrt{3}\,y\,(y + 3\sqrt{3}) - 3(y + 3\sqrt{3}) = 0$

$\Rightarrow (y + 3\sqrt{3}) - (\sqrt{3}\,y - 3) = 0$

$y \neq -3\sqrt{3}$

$\therefore\ y = \sqrt{3}$ as $y \in (0, 3)$

Case – II: $\frac{6y}{9-y^2} < 0 \Rightarrow y < 0$

$$\tan^{-1}\left(\frac{6y}{9-y^2}\right) + \pi + \tan^{-1}\left(\frac{6y}{9-y^2}\right) = \frac{2\pi}{3}$$

$$\Rightarrow 2\tan^{-1}\left(\frac{6y}{9-y^2}\right) = -\frac{\pi}{3} \Rightarrow \tan^{-1}\left(\frac{6y}{9-y^2}\right) = \frac{\pi}{6}$$

$$\Rightarrow \frac{6y}{9-y^2} = -\frac{1}{\sqrt{3}}$$

$$\Rightarrow 6\sqrt{3}\,y = -9 + y^2$$

$$\Rightarrow y^2 - 6\sqrt{3}\,y - 9 = 0$$

$$\Rightarrow y = \frac{6\sqrt{3} \pm \sqrt{108+36}}{2} = \frac{6\sqrt{3} \pm 12}{2} = 3\sqrt{3} \pm 6 \text{ as } y \in (-3.\ 0)$$

$\therefore\ y = 3\sqrt{3} - 6$

$\therefore$ Sum of solutions = $\sqrt{3} + (3\sqrt{3} - 6) = 4\sqrt{3} - 6$

Q.4 Let the position vectors of the points P,Q,R and S be $\vec{a} = \hat{i} + 2\hat{j} - 5\hat{k}, \vec{b} = 3\hat{i} + 6\hat{j} + 3\hat{k}$, $\vec{c} = \frac{17}{5}\hat{i} + \frac{16}{5}\hat{j} + 7\hat{k}$ and $\vec{b} = 2\hat{i} + \hat{j} + \hat{k}$, respectively. Then which of the following statements is true?

(A) The points P, Q, R and S are NOT coplanar

(B) $\frac{\vec{b}+2\vec{d}}{3}$ is the position vector of a point which divides PR internally in the ratio 5 : 4

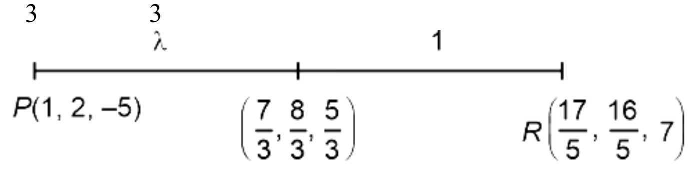
(C) $\frac{\vec{b}+2\vec{d}}{3}$ is the position vector of a point which divides PR externally in the ratio 5 : 4

(D) The square of the magnitude of the vector $\vec{b} \times \vec{d}$ is 95

Ans. **[B]**

Sol. $P(1,2,-5)$, $Q(3, 6, 3)$, $R\left(\frac{17}{5}, \frac{16}{5}, 7\right)$, $S(2, 1, 1)$

$$\frac{\vec{b}+2\vec{d}}{3} = \frac{7i+8j+5k}{3}$$



$$\Rightarrow \frac{17\lambda}{5} + 1 = \frac{7}{3}(\lambda + 1)$$

$$\Rightarrow 51\lambda + 15 = 35\lambda + 35$$

$$\Rightarrow 16\lambda = 20 \Rightarrow \lambda = \frac{5}{4}$$

SECTION – 2 (Maximum Mark : 12)

- This section contains **THREE (03)** questions
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	: **+4**	**ONLY** If (all) the correct option(s) is (are) chosen.
Partial Marks	: **+3**	If all the four options are correct but **ONLY** three options are chosen.
Partial Marks	: **+2**	If three or more options are correct but **ONLY** two options are chosen, both of which are correct.
Partial Marks	: **+1**	If two or more options are correct but **ONLY** one option is chosen and it is a correct option.
Zero Marks	: **0**	If unanswered;
Negative Marks	: **–2**	In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, than
choosing ONLY (A), (B) and (D) will get +4 marks;
choosing ONLY (A) and (B) will get +2 marks;
choosing ONLY (A) and (D) will get +2 marks;
choosing ONLY (B) and (D) will get +2 marks;
choosing ONLY (A) will get +1 mark;
choosing ONLY (B) will get +1 mark;
choosing ONLY (D) will get +1 mark;
choosing no option(s) (i.e. the question is unanswered) will get 0 marks and choosing any other option(s) will get –2 marks.

Q.5 Let M = (a_{ij}), i, j $\in$ {1,2,3}, be the 3 × 3 matrix such that a_{ij} = 1 if j + 1 is divisible by i, otherwise a_{ij} = 0. Then which of the following statements is(are) true?

(A) M is invertible

(B) There exists a nonzero column matrix $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ such that M $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} -a_1 \\ -a_2 \\ -a_3 \end{pmatrix}$

(C) The set $\{X \in R^3 : MX = 0\} \neq \{0\}$, where $0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

(D) The matrix (M – 2I) is invertible, where I is the 3 × 3 identity matrix

Ans. **[B, C]**

Sol. M = (a_{ij}), i, j $\in$ {1, 2, 3},

a_{ij} = 1 if j + 1 is divisible by i otherwise a_{ij} = 0 $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$|M| = 1(-1) - 1(-1) = -1 + 1 = 0$

M is not invertible

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \end{bmatrix}$$

$$\begin{bmatrix} a_1+a_2+a_3 \\ a_1+a_3 \\ a_2 \end{bmatrix} = \begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \end{bmatrix}$$

$a_1 + a_2 + a_3 = -a_1$ $\quad$ $a_1 + a_3 = -a_2$ $\quad$ $a_2 = -a_3$

$a_1 + a_2 + a_3 = 0$ $\quad$ $a_2 + a_3 = 0$

$a_1 = 0$ & $a_2 + a_3 = 0$

⇒ There exist a column matrix (infinite Possibilities)

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x+y+z=0 \\ x+z=0 \\ y=0 \end{matrix}$$ Yes it is possible

$$|M-2I| = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} = -1(3) - 1(-2-1) = -3+3 = 0$$

Q.6 Let f: (0,1) → R be the function defined as $f(x) = [4x]\left(x-\frac{1}{4}\right)^2\left(x-\frac{1}{2}\right)$, where [x] denotes the greatest integer less than or equal to x. Then which of the following statements is (are) true?

(A) The function f is discontinuous exactly at one point in (0, 1)

(B) There is exactly one point in (0, 1) at which the function f is continuous but NOT differentiable

(C) The function f is NOT differentiable at more than three points in (0, 1)

(D) The minimum value of the function f is $-\frac{1}{512}$

Ans. **[A, B]**

Sol. $f : (0, 1) \to R$

$f(x) = [4x]\left(x-\frac{1}{4}\right)^2\left(x-\frac{1}{2}\right) \Rightarrow$ Critical point = $= \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$

Discontinuity at $x = \frac{3}{4}$

Continuous and differentiable at $x = \frac{1}{4}$

Continuous but non-differentiable at $x = \frac{1}{2}$

$\text{LHD}\left(\text{at } x = \frac{1}{4}\right)$ $\qquad$ $\text{RHD}\left(\text{at } x = \frac{1}{4}\right)$

$\lim_{h \to 0^+} \frac{0-0}{-h} = 0$ $\qquad$ $\lim_{h \to 0^+} \frac{h^2\left(-\frac{1}{2}+h\right)}{h} = 0$

$\text{LHD}\left(\text{at } x = \frac{1}{2}\right)$ $\qquad$ $\text{RHD}\left(\text{at } x = \frac{1}{2}\right)$

$$\lim_{h \to 0^+} \frac{\left(\frac{1}{4}-h\right)^2(-h)-0}{-h} = \frac{1}{16}$$

$$\lim_{h \to 0^+} \frac{2\left(\frac{1}{4}+h\right)^2 h - 0}{h} = \frac{1}{8}$$

Minimum –ve value will exist between $\frac{1}{4}$ & $\frac{1}{2}$

$$f(x) = \left(x-\frac{1}{4}\right)^2\left(x-\frac{1}{2}\right) \qquad \frac{1}{4} \le x \le \frac{1}{2}$$

$$f'(x) = \left(x-\frac{1}{4}\right)\left(3x-\frac{5}{4}\right) \qquad \Rightarrow \text{minima at } x = \frac{5}{12}$$

$$f\left(\frac{5}{12}\right) = \frac{1}{36}\times\frac{-1}{12} = \frac{-1}{432}$$

Q.7 Let S be the set of all twice differentiable functions f from R to R such that $\frac{d^2f}{dx^2}(x) > 0$ for all $x \in (-1,1)$. For $f \in S$, let X_f be the number of points $x \in (-1, 1)$ for which $f(x) = x$. Then which of the following statements is (are) true?

(A) There exists a function $f \in S$ such that $X_f = 0$

(B) For every function $f \in S$, we have $X_f \le 2$

(C) There exists a function $f \in S$ such that $X_f = 2$

(D) There does NOT exist any function f in S such that $X_f = 1$

Ans. **[A, B, C]**

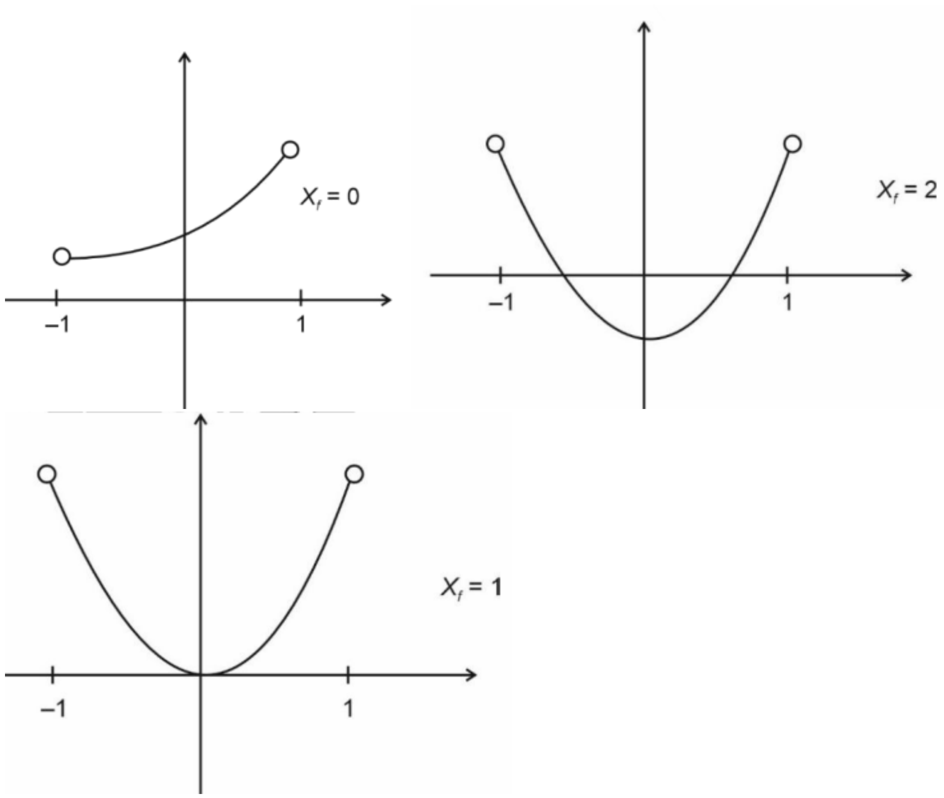
Sol. $f''(x) > 0$; $f(x) - x = 0$

Number of solutions = ?

Let $g(x) = f(x) - x \Rightarrow g'(x) = f'(x) - 1$

$g''(x) = f'(x) > 0 \Rightarrow$ Concave

Possibilities



SECTION – 3 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- The answer to each question is a **NON-NEGATIVE INTERGER.**
- Four each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	: **+4**	If **ONLY** the correct integer is entered;
Zero Marks	: **0**	In all other cases.

Q.8 For $x \in R$, let tan–1 (x) $\in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the minimum value of the function $f : R \to R$ defined by

$$f(x) = \int_0^{x\tan^{-1}x} \frac{e^{(t-\cos t)}}{1+t^{2023}}dt \text{ is}$$

Ans. **[0]**

Sol. $$f'(x) = \frac{e^{[x\tan^{-1}x - \cos(x\tan^{-1}x)]}}{1+(x\tan^{-1}x)^{2023}} \times \left(\frac{x}{1+x^2} + \tan^{-1}x\right)$$

$f'(x) = g(x).h(x)$

where $g(x) = \frac{e^{[x\tan^{-1}x - \cos(x\tan^{-1}x)]}}{1+(x\tan^{-1}x)^{2023}} > 0 \ \forall \ x$

and $h(x) = \frac{x}{1+x^2} + \tan^{-1}x < 0$ for $x < 0$

$= 0 \ x = 0 \ > 0 \ x > 0$

$\therefore$ f(x) has minimum at $x = 0$

And $f(x)_{min} = f(0) = 0$

Q.9 For $x \in R$, let y(x) be a solution of the differential equation

$(x^2-5)\frac{dy}{dx} - 2xy = -2x(x^2-5)^2$ such that $y(2) = 7$.

Then the maximum value of the function y(x) is

Ans. **[16]**

Sol. $$(x^2-5)\frac{dy}{dx} - 2xy = -2x(x^2-5)^2$$

$$\frac{dy}{dx} + \left(\frac{-2x}{x^2-5}\right)y = -2x(x^2-5)$$

$$\text{I.F.} = \frac{1}{|x^2-5|}$$

Solution of D. E. is $y.\frac{1}{|x^2-5|} = \int -2x.\frac{x^2-5}{|x^2-5|}dx \Rightarrow \frac{y}{|x^2-5|} = \frac{x^2-5}{|x^2-5|}(-x^2) + C$

$\therefore y(2) = 7 \Rightarrow C = 3$

$\Rightarrow y = -x^2(x^2-5) + 3|x^2-5| \Rightarrow y = f(x)$ is even function

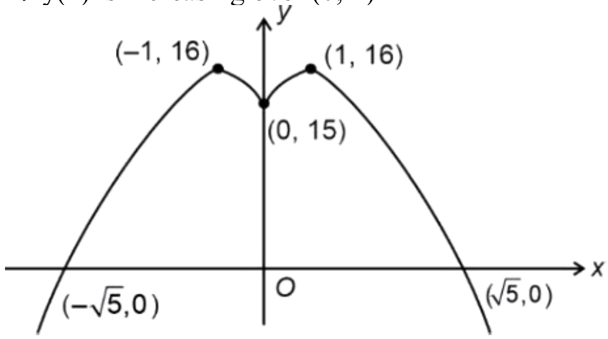
If $0 < x < \sqrt{5}$, $y = -x^4 + 5x^2 - 3x^2 + 15 = -x^4 + 2x^2 + 15$

For increasing function $\frac{dy}{dx} > 0 \Rightarrow x < 1$

If $x > \sqrt{5}$, $y = -x^4 + 5x^2 + 3x^2 - 15$

For increasing function $\frac{dy}{dx} > 0 \Rightarrow x = \phi$

$\Rightarrow$ y(x) is increasing over (0, 1)



$\Rightarrow f(x)_{max} = 16$

Q.10 Let X be the set of all five digit numbers formed using 1,2,2,2,4,4,0. For example, 22240 is in X while 02244 and 44422 are not in X. Suppose that each element of X has an equal chance of being chosen. Let p be the conditional probability that an element chosen at random is a multiple of 20 given that it is a multiple of 5. Then the value of 38p is equal to

Ans. **[31]**

Sol. Number of five-digit numbers divisible by 5 (__________0)

_ _ _ _ 0

$2224 \longrightarrow 4$

$2244 \longrightarrow 6$

$2221 \longrightarrow 4$

$2241 \longrightarrow 12$

$2441 \longrightarrow \frac{12}{38}$

Number of five-digit numbers divisible by 5 but' not' by 20

_ _ _ 1 0

$222 \longrightarrow 1$

$224 \longrightarrow 3$

$244 \longrightarrow \frac{3}{7}$

Number of five-digit numbers divisible by 5 'and' 20 = 38 – 7 = 31

$P = \frac{31}{38}$

38p = 31

Q.11 Let $A_1, A_2, A_3, ..., A_8$ be the vertices of a regular octagon that lie on a circle of radius 2. Let P be a point on the circle and let PA_i denote the distance between the points P and A_i for i = 1,2,....,8. If P varies over the circle, then the maximum value of the $PA_1.PA_2.....PA_8$, is

Ans. **[512]**

Sol. $A_1, A_2, A_3, ..., A_8$ vertices of a regular octagon lying on a circle of radius 2.

Let say, $Z = (2)(1)^{1/8}$

$\Rightarrow Z^8 = 2^8 \times 1$

$\Rightarrow Z^8 - 2^8 = 0$

$\Rightarrow Z = 2, 2\alpha, 2\alpha^2, 2\alpha^3, ..., 2\alpha^7; \alpha = e^{i\frac{2\pi}{8}}$

$\Rightarrow Z^8 - 2^8 = (Z - 2)(Z - 2\alpha)(Z - 2\alpha^2)(Z - 2\alpha^3) ....(Z - 2\alpha^7)$

$\Rightarrow |Z^8 - 2^8| = |Z - 2||Z - 2\alpha| ....|Z - 2\alpha^7|$

But $|Z^8 + (-2^8)| \le |Z|^8 + 2^8$

$\Rightarrow |Z - 2|\,|Z - 2\alpha| ...|Z - 2\alpha^7| \le |Z|^8 + 2^8$

$\le 2^8 + 2^8$

$\le 2^9$

$\Rightarrow$ Max $(PA_1.PA_2....PA_8) = 2^9$

Q.12 Let $R = \left\{\begin{pmatrix} \alpha & 3 & b \\ c & 2 & d \\ 0 & 5 & 0 \end{pmatrix} : a, b, c, d \in \{0,3,5,7,11,13,17,19\}\right\}$. Then the number of invertible matrices in R is

Ans. **[3780]**

Sol. $|R| = -5\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

|R| can be zero in following cases:

(i) Two of a, b, c, d are zeroes which can be (a and b), (b are d), (d and c) or (c and a)

$\rightarrow 4 \times 7^2$ ways = 196

(ii) Any three of a, b, c, d are zeroes

$\rightarrow {}^4C_3 \times 7 = 28$

(iii) All four of a, b, c, d are zeroes

$\rightarrow$ 1

(iv) All four of a, b, c, d are non –zero but same number

$\rightarrow$ 7

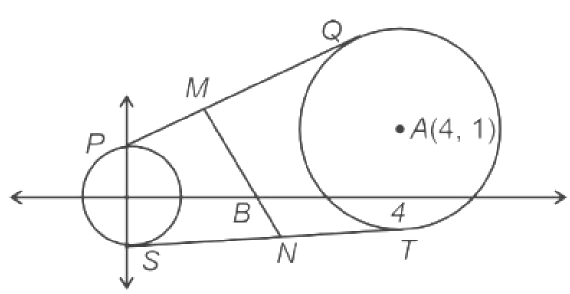
(v) When two are alike and 2 other are alike (non-zero) $\rightarrow 7C_2 \times 2 \times 2 = 84$

Number of invertible matrices $= 8^4 - 196 - 28 - 1 - 7 - 84 = 3780$

Q.13 Let C_1 be the circle of radius 1 with center at the origin. Let C_2 be the circle of radius r with center at the point A = (4,1), where $1 < r < 3$. Two distinct common tangents PQ and ST of C_1 and C_2 are drawn. The tangent PQ touches C_1 at P and C_2 at Q. The tangent ST touches C_1 at S and C_2 at T. Mid points of the line segments PQ and ST are joined to form a line which meets the x-axis at a point B. If $AB = \sqrt{5}$, then the value of r^2 is

Ans. **[2]**

Sol.



Let M and N be midpoints of PQ and ST respectively.

$\Rightarrow$ MN is a radical axis of two circles

$C_1 : x^2 + y^2 = 1$ ... (i)

$C_2 : (x - 4)^2 + (y - 1)^2 = r^2$

$\Rightarrow x^2 + y^2 - 8x - 2y + 17 - r^2 = 0$ ... (ii)

From (i) and (ii);

Equation of MN : $8x + 2y - 18 + r^2 = 0$

$\Rightarrow$ B is on x-axis $\Rightarrow$ B $\left(\frac{18-r^2}{8}, 0\right)$

AB = $\sqrt{5}$

$\sqrt{\left(\frac{18-r^2}{8} - 4\right)^2 + 1} = \sqrt{5}$ (By distance formed a)

$\Rightarrow$ On solving $r^2 = 2$

SECTION – 4 (Maximum Marks: 12)

- This section contains **TWO (02)** paragraphs.
- Based on each paragraph, there are TWO (02) questions.
- The answer to each question is a **NUMERICAL VALUE.**
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	: **+3**	f **ONLY** the correct numerical value is entered in the designated place;
Zero Marks	: **0**	In all other cases.

PARAGRAPH"I" (Q.14 to Q.15)

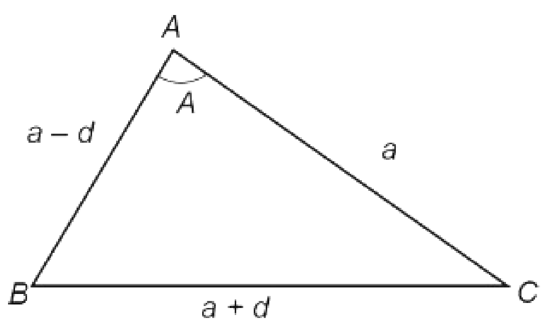
Consider an obtuse angled triangle ABC in which the difference between the largest and the smallest angle is $\frac{\pi}{2}$ and whose sides are in arithmetic progression. Suppose that the vertices of this triangle lie on a circle of radius 1.

(There are two questions based on PARAGRAPH "I", the question given below is one of them)

Q.14 Let a be the area of the triangle ABC. Then the value of $(64a)^2$ is

Ans. **[1008]**

Sol.



Let sides be a – d, a, a + d

$A - C = \frac{\pi}{2}$

R = 1

Now

$\frac{a+d}{\sin A} = \frac{a}{\sin B} = \frac{a-d}{\sin C} = 2$

$\therefore\ A = \frac{\pi}{2} + C$

$\sin A = \sin\left(\frac{\pi}{2} + C\right)$

sin A = cos C.

$\frac{a+d}{2} = \sqrt{1-\sin^2 C}$

$\left(\frac{a+d}{2}\right)^2 = 1 - \left(\frac{a-d}{2}\right)^2$

$\frac{2(a^2+d^2)}{4} = 1$

$a^2 + d^2 = 2$ ....(1)

Now $\cos B = \frac{(a-d)^2 + (a+d)^2 - a^2}{2(a^2-d^2)}$

$\sqrt{1-\sin^2 B} = \frac{2(a^2+d^2) - a^2}{2(a^2-d^2)}$

$\sqrt{1-\frac{a^2}{4}} = \frac{4-a^2}{2(a^2-d^2)}$ $(\because a^2 + d^2 = 2)$

$(a^2-d^2)^2 = 4 - a^2$

From (1) & (2)

$a^2 = \frac{7}{4}, d^2 = \frac{1}{4}$ ....(2)

Area of triangle

$\Delta = \frac{a(a^2-d^2)}{4}$

$\alpha = \frac{\sqrt{7}}{2} \times \frac{6}{4\times 4}$

$(64\ \alpha)^2 = 1008$

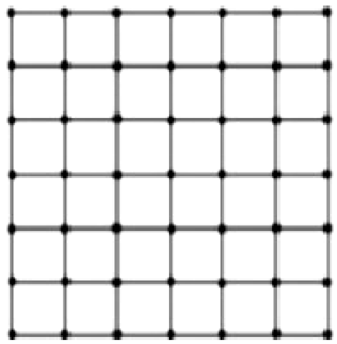
Q.15 Then the inradius of the triangle ABC is

Ans. **[0.25]**

Sol. $r = \frac{\Delta}{S} = \frac{\frac{\sqrt{7}}{2}\times\frac{6}{16}}{\frac{3}{2}\times\frac{\sqrt{7}}{2}} = \frac{4}{16} = \frac{1}{4} = 0.25$

PARAGRAPH "II"(Q.16 to Q.17)

Consider the 6 × 6 square in the figure. Let $A_1, A_2, ..., A_{49}$ be the points of intersections (dots in the picture) in some order. We say that A_i and A_j are friends if they are adjacent along a row or along a column. Assume that each point A_i has an equal chance of being chosen.

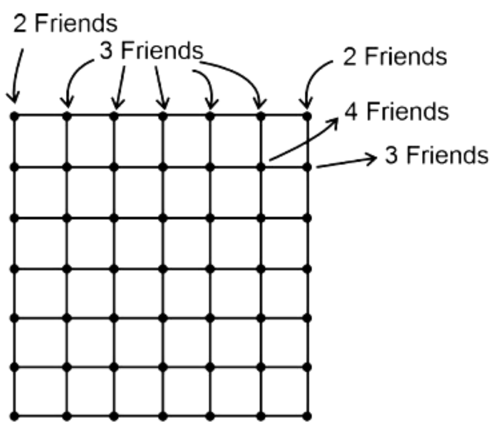


(There are two questions based on PARAGRAPH "II", the question given below is one of them)

Q.16 Let P_i be the probability that a randomly chosen point has i many friends, i = 0,1,2,3,4. Let X be a random variable such that for i = 0,1,2,3,4, the probability $P(X = i) = p_i$. Then the value of 7E(X) is

Ans. **[24]**

Sol.



Number of points having 0 friend = 0
Number of points having 1 friend = 0
Number of points having 2 friends = 4
Number of points having 3 friends = 5 × 4 = 20
Number of points having 4 friends = 49 – 24 = 25

P_i	0	0	4/49	20/49	25/49
X_i	0	1	2	3	4

$$7\,(E(X)) = 7\left(0 + 0 + \frac{4}{49} \times 2 + \frac{20}{49} \times 3 + \frac{25}{49} \times 4\right)$$

$$= \left(\frac{100 + 60 + 8}{49}\right)$$

= 24

Q.17 Two distinct points are chosen randomly out of the points $A_1, A_2, ..., A_{49}$. Let P be the probability that they are friends. Then the value of 7 P is

Ans. [**0.50**]

Sol. Number of ways of selecting 2 adjacent dots in /row = 6

Similarly, number of ways of selecting 2 adjacent dots in I column = 6

$\therefore$ Number of ways of selecting 2 adjacent dots from the matrix = 6 × 7 + 6 × 7 = 84

$\therefore P = \frac{84}{^{49}C_2} = \frac{84 \times 2}{49 \times 48}$

$7p = \frac{7 \times 84 \times 2}{49 \times 48} = \frac{1}{2}$

0.50 is answer

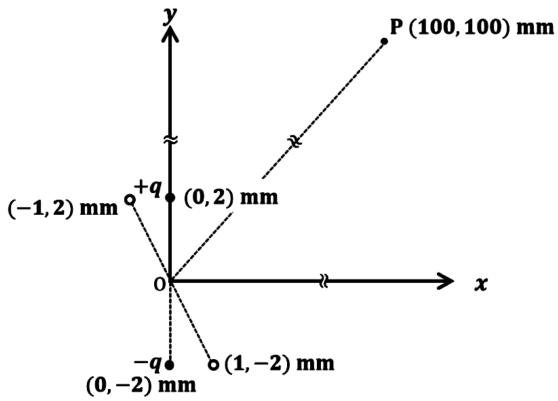
PHYSICS

SECTION – 1 (Maximum Mark : 12)

- This section contains **FOUR (04)** questions.
- Each question has four options (A), (B), (C) and (D). **Only one** of these four option is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	: **+3**	If **ONLY** the correct option is chosen;
Zero Marks	: **0**	If none of the option is chosen (i.e. the question is unanswered);
Negative Marks	: –1	In all other cases.

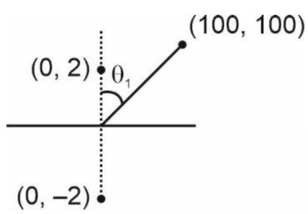
Q.1 An electric dipole is formed by two charges +q and –q located in xy-plane at (0,2) mm and (0, –2) mm, respectively, as shown in the figure. The electric potential at point P (100, 100) mm due to the dipole is V_0. The charges +q and –q are then moved to the points (–1, 2) mm and (1, –2) mm, respectively. What is the value of electric potential at P due to the new dipole?



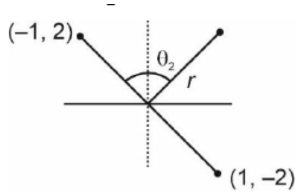
(A) $V_0/4$ (B) $V_0/2$ (C) $V_0/\sqrt{2}$ (D) $3V_0/4$

Ans. **[B]**

Sol. $V_1 \propto \dfrac{p_1 \cos\theta_1}{r_1^3}$



$V_2 \propto \dfrac{p_2 \cos\theta_2}{r_2^3}$



$$\frac{V_2}{V_1} = \frac{p_2 \cos\theta_2}{p_1 \cos\theta_1}$$

$$\frac{V_2}{V_1} = \frac{q(-2\hat{i}+4\hat{j}).(\hat{i}+\hat{j})}{q(0\hat{i}+4\hat{j}).(\hat{i}+\hat{j})} = \frac{1}{2}$$

$$\Rightarrow V_2 = \frac{V_0}{2}$$

Q.2 Young's modulus of elasticity Y is expressed in terms of three derived quantities, namely the gravitational constant G, Plank's constant h and the speed of light c, as $Y = c^{\alpha}h^{\beta}G^{\gamma}$. Which of the following is the correct option?

(A) $\alpha = 7, \beta = -1, \gamma = -2$ (B) $\alpha = -7, \beta = -1, \gamma = -2$

(C) $\alpha = 7, \beta = -1, \gamma = 2$ (D) $\alpha = -7, \beta = 1, \gamma = -2$

Ans. **[A]**

Sol. $Y = c^{\alpha}h^{\beta}G^{\gamma}$

$[M^1L^{-1}T^{-2}] = [M^0L^1T^{-1}]^{\alpha}\,[M^1L^2T^{-1}]^{\beta}\,[M^{-1}L^3T^{-2}]^{\gamma}$

$1 = \beta - \gamma$

$-1 = \alpha + 2\beta + 3\gamma$

$-2 = -\alpha - \beta - 2\gamma$

Solving

$\alpha = 7, \beta = -1, \gamma = -2$

Q.3 A particle of mass m is moving in the xy-plane such that its velocity at a point (x, y) is given as $\vec{v} = \alpha(y\hat{x} + 2x\hat{y})$, where α is a non-zero constant. What is the force $\vec{F}$ acting on the particle?

(A) $\vec{F} = 2m\alpha^2(x\hat{x} + y\hat{y})$ (B) $\vec{F} = m\alpha^2(y\hat{x} + 2x\hat{y})$ (C) $\vec{F} = 2m\alpha^2(y\hat{x} + x\hat{y})$ (D) $\vec{F} = m\alpha^2(x\hat{x} + 2y\hat{y})$

Ans. **[A]**

Sol. $F = m\dfrac{d\vec{v}}{dt}$

$\vec{v} = \alpha(y\hat{x} + 2x\hat{y})$

$$\frac{d\vec{v}}{dt} = \alpha\left(\frac{dy}{dt}\hat{x} + 2\frac{dx}{dt}\hat{y}\right)$$

$= \alpha(v_y\hat{x} + 2v_x\hat{y})$

$= \alpha[2x\alpha\hat{x} + 2\alpha y\hat{y}]$

$= 2\alpha^2[x\hat{x} + y\hat{y}]$

$\vec{F} = 2m\alpha^2[x\hat{x} + y\hat{y}]$

Q.4 An ideal gas is in thermodynamic equilibrium. The number of degrees of freedom of a molecule of the gas is n. The internal energy of one mole of the gas is U_n and the speed of sound in the gas is v_n. At a fixed temperature and pressure, which of the following is the correct option?

(A) $v_3 < v_6$ and $U_3 > U_6$ (B) $v_5 > v_3$ and $U_3 > U_5$ (C) $v_5 > v_7$ and $U_5 < U_7$ (D) $v_6 < v_7$ and $U_6 < U_7$

Ans. **[C]**

Sol. $U_n = \dfrac{1 \times n \times RT}{2} = \dfrac{nRT}{2}$

$$v_n = \sqrt{\frac{\gamma RT}{M}} = \sqrt{\frac{\left(1 + \frac{2}{n}\right)RT}{M}}$$

$\Rightarrow U_7 > U_5$ and $U_7 > U_6$

and $v_5 > v_7$

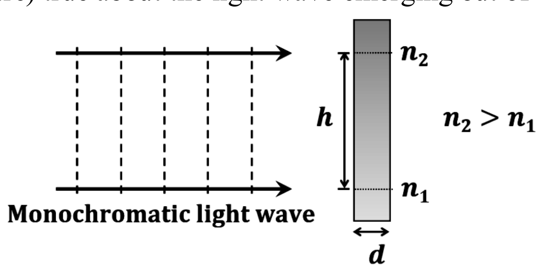
SECTION – 2 (Maximum Mark : 12)

- This section contains **THREE (03)** questions
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	: **+4**	**ONLY** If (all) the correct option(s) is (are) chosen.
Partial Marks	: **+3**	If all the four options are correct but **ONLY** three options are chosen.
Partial Marks	: **+2**	If three or more options are correct but **ONLY** two options are chosen, both of which are correct.
Partial Marks	: **+1**	If two or more options are correct but **ONLY** one option is chosen and it is a correct option.
Zero Marks	: **0**	If unanswered;
Negative Marks	: **–2**	In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, than
  choosing ONLY (A), (B) and (D) will get +4 marks;
  choosing ONLY (A) and (B) will get +2 marks;
  choosing ONLY (A) and (D) will get +2 marks;
  choosing ONLY (B) and (D) will get +2 marks;
  choosing ONLY (A) will get +1 mark;
  choosing ONLY (B) will get +1 mark;
  choosing ONLY (D) will get +1 mark;
  choosing no option(s) (i.e. the question is unanswered) will get 0 marks and choosing any other option(s) will get –2 marks.

Q.5 A monochromatic light wave is incident normally on a glass slab of thickness d, as shown in the figure. The refractive index of the slab increases linearly from n_1 to n_2 over the height h. Which of the following statement(s) is(are) true about the light wave emerging out of the slab?



(A) It will deflect up by an angle $\tan^{-1}\left[\frac{(n_2^2 - n_1^2)d}{2h}\right]$

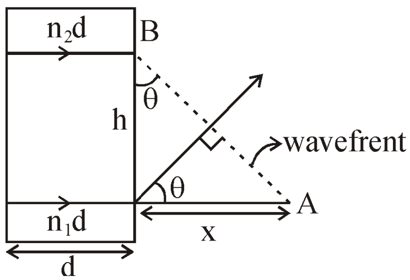
(B) It will deflect up by an angle $\tan^{-1}\left[\frac{(n_2 - n_1)d}{h}\right]$

(C) It will not deflect.

(D) The deflection angle depends only on $(n_2 - n_1)$ and not the individual values of n_1 and n_2.

Ans. **[B.D]**

Sol.



Phase at A and B should be same, so equating optical path :

$n_2d = n_1d + x$

$\Rightarrow x = (n_2 - n_1)d$

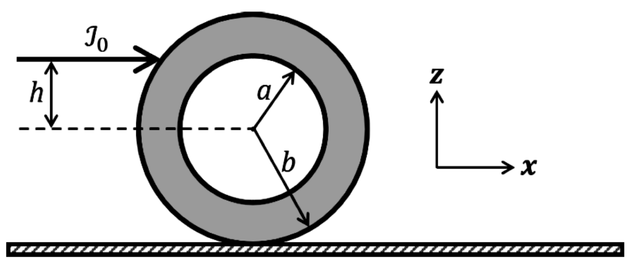
$\tan\theta = \frac{x}{h} = \frac{(n_2 - n_1)d}{h} = \tan^{-1}\left\{\frac{(n_2 - n_1)d}{h}\right\}$

(B) is correct

θ depends upon $(n_2 - n_1)$

So (D) is also correct

Q.6 An annular disk of mass M, inner radius a and other radius b is placed on a horizontal surface with coefficient on friction μ, as shown in figure. At some time, an impulse $J_0\hat{x}$ is applied at a height h above the center of the disk. If $h = h_m$ then the disk rolls without slipping along the x-axis. Which of the following statement(s) is(are) correct?



(A) For $\mu \neq 0$ and $a \to 0$, $h_m = b/2$.

(B) For $\mu \neq 0$ and $a \to 2$, $h_m = b$.

(C) For $h = h_m$, the initial angular velocity does not depend on the inner radius a.

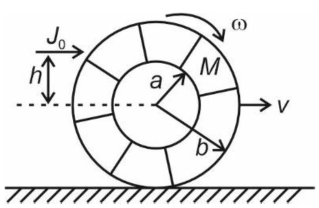
(D) For $\mu = 0$ and $h = 0$, the wheel slides without rolling.

Ans. **[A,B,C,D]**

Sol. $\vec{J} = \overrightarrow{\Delta p}$

$\Rightarrow J_0 = MV$ ...(i)

About



$\overrightarrow{A.j} = \overrightarrow{\Delta L}$

$\Rightarrow J_0 \times h = \frac{M}{2}(a^2 + b^2)\omega$ ...(i)

If $h = h_m \Rightarrow \omega = \frac{v}{b}$

$\Rightarrow J_0 \times h_m = \left(\frac{m}{2}\right)(a^2 + b^2)\ \frac{v}{b}$ ...(ii)

Equation (i) ÷ Equation (ii)

$\Rightarrow h_m = \frac{a^2 + b^2}{2b}$

If $a \to 0 \Rightarrow h_m = b/2$

If $a \to b \Rightarrow h_m = b$

If $h = h_m \Rightarrow \omega$ is independent of a (equation A)

If $r = 0$, $h = 0 \Rightarrow$ always sliding

Q.7 The electric field associated with an electromagnetic wave propagating in a dielectric medium is given by $\vec{E} = 30(2\hat{x} + \hat{y})\sin\left[2\pi\left(5\times10^{14}t - \frac{10^7}{3}z\right)\right]$ Vm^{-1}. Which of the following option(s) is (are) correct?

[Given: The speed of light in vacuum, $c = 3 \times 10^8$ m s^{-1}]

(A) $B_x = -2 \times 10^{-7} \sin\left[2\pi\left(5\times10^{14}t - \frac{10^7}{3}z\right)\right]$ Wb m^{-2}

(B) $B_y = 2 \times 10^{-7} \sin\left[2\pi\left(5\times10^{14}t - \frac{10^7}{3}z\right)\right]$ Wb m^{-2}

(C) The wave is polarized in the xy-plane with polarization angle 30^0 with respect to the x-axis.

(D) The refractive index of the medium is 2.

Ans. **[A,D]**

Sol. Speed of light in medium is $V = \omega/k$

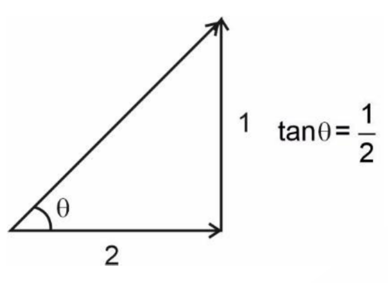
$V = \frac{3\times5\times10^{14}}{10^7}$

$V = 1.5 \times 10^8$

Refractive index $\mu = \frac{C}{V} = \frac{3\times10^8}{1.5\times10^8} = 2$

$\mu = 2$

Given $\vec{E} = 30(2\hat{x} + \hat{y})\sin\left[2\pi\left(5\times10^{14}t - \frac{10^7}{3}2\right)\right]^{1/m}$



$B_0 = \frac{E_0}{V} = \frac{30\sqrt{5}}{1.5\times10^8}$

Direction of $\vec{B}_0$ is $(\vec{v}\times\vec{E})$

$\vec{v}\times\vec{E} = \hat{k}\times\frac{(2\hat{i}+\hat{j})}{\sqrt{5}}$

$\left(\frac{-\hat{i}+2\hat{j}}{\sqrt{5}}\right)$ put value $\vec{B}_0 = \frac{30\sqrt{5}}{1.5\times10^8}\times\left(\frac{-\hat{i}+2\hat{j}}{\sqrt{5}}\right)$

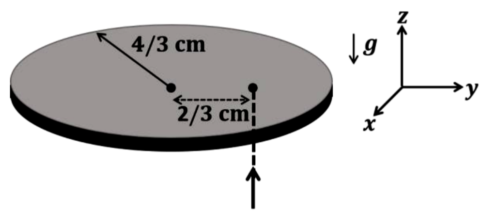
$B_x = -2 \times 10^7$

SECTION – 3 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- The answer to each question is a **NON-NEGATIVE INTERGER.**
- Four each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	: **+4**	If **ONLY** the correct integer is entered;
Zero Marks	: **0**	In all other cases.

Q.8 A thin circular coin of mass 5 gm and radius 4/3 cm is initially in a horizontal xy-plane. The coin is tossed vertically up (+z direction) by applying an impulse of $\sqrt{\frac{\pi}{2}}\times10^{-2}$ N-s at a distance 2/3 cm from its center. The coin spins about its diameter and moves along the +z direction. By the time the coin reaches back to its initial position, it completes n rotations. The value of n is ____.
[Given: The acceleration due to gravity g = 10 m s^{-2}]



Ans. **[30]**

Sol. By impulse – momentum theorem :

$J = MV_{CM}$

$$\Rightarrow V_{CM} = \frac{J}{M} = \frac{\sqrt{\frac{\pi}{2}}}{100\times\frac{5}{1000}} = \sqrt{2\pi} \text{ m/s}$$

$$\Rightarrow \text{Total time of journey} = \frac{2}{g} \times \sqrt{2\pi}$$

$$\Rightarrow \Delta t = \frac{\sqrt{2\pi}}{5} \text{sec}$$

Also, by angular impulse – momentum theorem:

$$J\times\frac{R}{2} = \left[\frac{MR^2}{4}\right]\omega$$

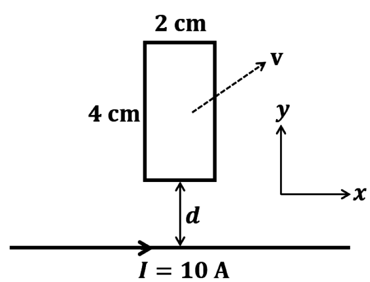
$$\Rightarrow \omega = \frac{J \times \frac{R}{2}}{\frac{MR^2}{4}} = \frac{J}{MR} \times 2 = \frac{\frac{\sqrt{\pi/2}}{100} \times 2}{\frac{5}{1000} \times \frac{4}{3} \times \frac{1}{100}}$$

$$= 2 \times 75\sqrt{2\pi} \text{ rad/s}$$

$$\Rightarrow \text{Number of rotations} = \frac{\omega.\Delta t}{2\pi} = \frac{2 \times 75\sqrt{2\pi} \times \frac{\sqrt{2\pi}}{5}}{2\pi} = 30$$

$\Rightarrow$ n = 30

Q.9 A rectangular conducting loop of length 4 cm and width 2 cm is in the xy-plane, as shown in the figure. It is being moved away from a thin and long conducting wire along the direction $\frac{\sqrt{3}}{2}\hat{x} + \frac{1}{2}\hat{y}$ with a constant speed v. The wire is carrying a steady current I = 10 A in the positive x-direction. A current of 10μA flows through the loop when it is at a distance d = 4 cm from the wire. If the resistance of the loop is 0.1 Ω, then the value of v is _______ m s^{-1}.
[Given: The permeability of free space □$\mu_0 = 4\pi \times 10^{-7}$ NA^{-2}]



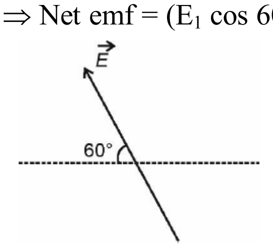
Ans. [4]

Sol. The two sides perpendicular to the wire would contribute net zero emt.
For parallel sides :

$\vec{E} = \vec{B} \times \vec{v}$

$= \frac{\mu_0 i}{2\pi x} \times v$

$\Rightarrow$ Net emf = (E_1 cos 60º – E_2 cos 60º) × width



$$= \frac{1}{2} \times \frac{2}{100} \times \frac{\mu_0 Iv}{2\pi}\left[\frac{1}{4/100} - \frac{1}{8/100}\right]$$

$$= \frac{1}{100} \times 10^{-7} \times 2 \times 10 \times v \times 100 \times \frac{1}{8}$$

$= 2.5v \times 10^{-7} = i \times R$

$$\Rightarrow v = \frac{10 \times 10^{-6} \times 0.1}{2.5 \times 10^{-7}} = 4 \text{ m/s}$$

Q.10 A string of length 1 m and mass 2×10^{-5} kg is under tension T. When the string vibrates, two successive harmonics are found to occur at frequencies 750 Hz and 1000 Hz. The value of tension T is _____ Newton.

Ans. **[5]**

Sol. $\ell = 1$m, m = 2×10^{-5} kg, T : Tension in the string.

$\because$ Successive frequencies are being given

$\therefore$ It is the case of both ends fixed.

Now, $f_{n+1} - f_n = 1000 - 750$

$$\Rightarrow \frac{(n+1)}{2\ell}\sqrt{\frac{T}{\mu}} - \frac{n}{2\ell}\sqrt{\frac{T}{\mu}} = 250$$

$$\Rightarrow \frac{1}{2\ell}\sqrt{\frac{T}{\mu}} = 250$$

$$\Rightarrow \sqrt{\frac{T}{2\times10^{-5}}} = 250 \times 2 \times 1$$

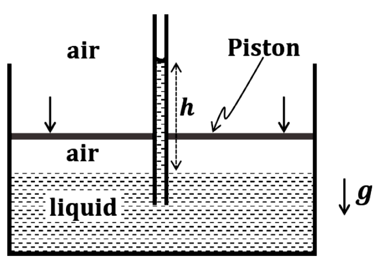
$$\Rightarrow \frac{T}{2\times10^{-5}} = 25 \times 10^{-4}$$

$\Rightarrow T = 50 \times 10^{-1}$

T = 5N

Q.11 An incompressible liquid is kept in a container having a weightless piston with a hole. A capillary tube of inner radius 0.1 mm is dipped vertically into the liquid through the airtight piston hole, as shown in the figure. The air in the container is isothermally compressed from its original volume V_0 to $\frac{100}{101}V_0$ with the movable piston. Considering air as an ideal gas, the height (h) of the liquid column in the capillary above the liquid level in cm is_______.

[Given: Surface tension of the liquid is 0.075 N m^{-1}, atmospheric pressure is 10^5 Nm^{-2}, acceleration due to gravity (g) is 10 ms^{-2}, density of the liquid is 10^3 kg m^{-3} and contact angle of capillary surface with the liquid is zero]



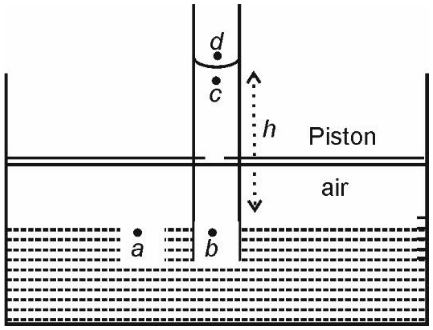
Ans. **[25]**

Sol. Let P_f be the air pressure

$P_0v_0 = P_fv_f$

$$P_0v_0 = P_f\left(\frac{100}{101}\right)v_0$$

$P_f = 101 \times 10^3$Pa $\qquad (\because P_0 = 10^5\ Nm^{-2})$



Now, consider the 4 points shown in diagram

$P_d - P_c = \frac{2T}{R}$ $\quad$ $(\because P_d = P_0)$

$\therefore P_c = P_0 - \frac{2T}{R}$

Now,

$P_a = P_b$ $\quad$ (also, $P_a = P_f$)

$P_f = \rho gh + P_c$

$101 \times 10^3 = (10^3 \times 10 \times h) + \left(10^5 - \frac{2\times 0.075}{0.1\times 10^{-3}}\right)$

$h = \frac{1}{4} m = 25$ cm

Q.12 In a radioactive decay process, the activity is defined as $A = -\frac{dN}{dt}$, where N(t) is the number of radioactive nuclei at time t. Two radioactive sources S_1 and S_2 have same activity at time t = 0. At a later time, the activities of S_1 and S_2 are A_1 and A_2, respectively. When S_1 and S_2 have just completed their 3^{rd} and 7^{th} half lives, respectively, the ratio A_1/A_2 is ____.

Ans. **[16]**

Sol. $A_1 = A_0 e^{-\lambda_1 t_1}$

also $A_2 = A_0 e^{-\lambda_2 t_2}$

at $t_1 = \frac{3\ln 2}{\lambda_1}$,

$A_1 = A_0 e^{-\lambda_1 \frac{3\ln 2}{\lambda_1}} = A_0 e^{-3\ln 2}$

Similarly, at

$t_2 = \frac{7\ln 2}{\lambda_2}$,

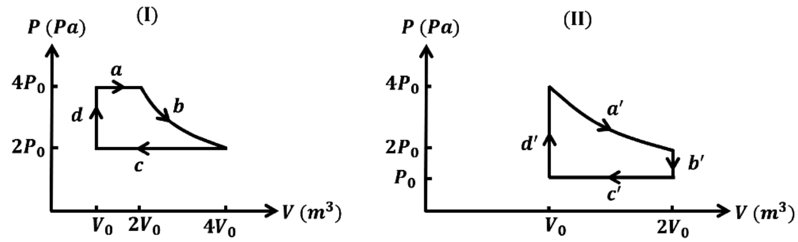
$A_2 = A_0 e^{-\lambda_2 \frac{7\ln 2}{\lambda_2}} = A_0 e^{-7\ln 2}$ ....(ii)

From (i) and (ii)

$\frac{A_1}{A_2} = \frac{A_0 e^{-3\ln 2}}{A_2 e^{-7\ln 2}} = \frac{2^{-3}}{2^{-7}} = \frac{1}{2^{-4}} = 2^4 = 16$

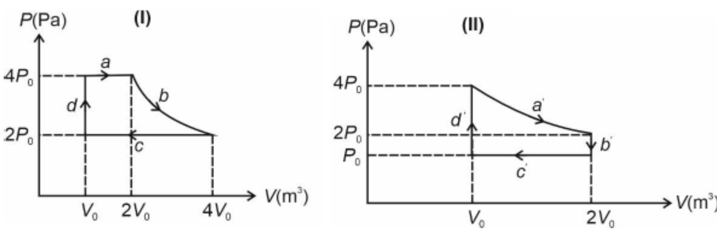
$\therefore \frac{A_1}{A_2} = 16$

Q.13 One mole of an ideal gas undergoes two different cyclic processes I and II, as shown in the P-V diagrams below. In cycle I, processes a, b, c and d are isobaric, isothermal, isobaric and isochoric, respectively. In cycle II, processes a', b', c' and d' are isothermal, isochoric, isobaric and isochoric, respectively. The total work done during cycle I is W_I and the during cycle II is W_{II}. The ratio W_I/W_{II} is _____.



Ans. **[2]**

Sol.



$W_I = W_a + W_b + W_c + W_d$

$= 4P_0(2V_0 - V_0) + nRT \ln\left(\frac{4V_0}{2V_0}\right) + 2P_0(V_0 - 4V_0) + 0$

$= 4P_0V_0 + nR\left(\frac{8P_0V_0}{nR}\right)\ln 2 - 6P_0V_0$

$= 8P_0V_0 \ln 2 - 2P_0V_0$

$W_n = W_a' + W_b' + W_c' + W_d'$

$= nRT \ln\left(\frac{2V_0}{V_0}\right) + 0 + P_0(V_0 - 2V_0) + 0$

$= nR\left(\frac{4P_0V_0}{nR}\right)\ln 2 - P_0V_0$

$= 4P_0V_0 \ln 2 - P_0V_0$

$\frac{W_I}{W_{II}} = \frac{8P_0V_0 \ln 2 - 2P_0V_0}{4P_0V_0 \ln 2 - P_0V_0} = 2$

SECTION – 4 (Maximum Marks: 12)

- This section contains **TWO (02)** paragraphs.
- Based on each paragraph, there are TWO (02) questions.
- The answer to each question is a **NUMERICAL VALUE.**
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme :

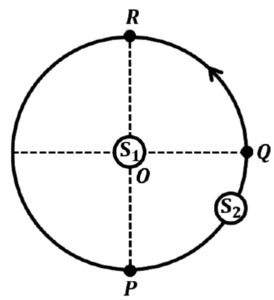
  Full Marks : **+3** f **ONLY** the correct numerical value is entered in the designated place;

  Zero Marks : **0** In all other cases.

PARAGRAPH I (Q.14 & 15)

S_1 and S_2 are two identical sound sources of frequency 656 Hz. The source S_1 is located at O and S_2 moves anti-clockwise with a uniform speed $4\sqrt{2}$ m s^{-1} on a circular path around O, as shown in the figure. There are three points P, Q and R on this path such that P and R are diametrically opposite while Q is equidistant from them. A sound detector is placed at point P. The source S_1 can move along direction OP.

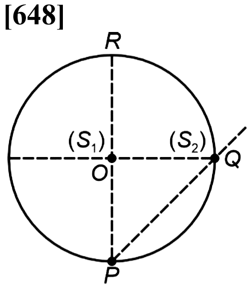
[Given: The speed of sound in air is 324 m s^{-1}]



Q.14 When only S_2 is emitting sound and it is at Q, the frequency of sound measured by the detector in Hz is ________.

Ans. **[648]**

Sol.



$f_0 = 656$ Hz

Velocity of sound = 324 m/s.

Velocity of source away from detector

$V_s = 4\sqrt{2}\ \cos 45^o = 4$ m/s

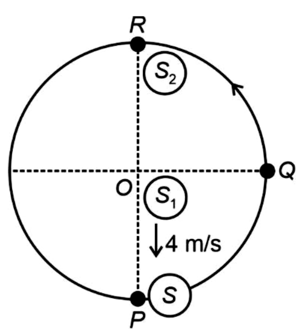
$\therefore\ f = \left(\frac{v}{v + v_s}\right) f_0 = \left(\frac{324}{324 + 4}\right) 656$

f = 648 Hz.

Q.15 Consider both sources emitting sound. When S_2 is at R and S_1 approaches the detector with a speed 4m s^{-1}, the beat frequency measured by the detector is _____Hz.

Ans. **[8.20]**

Sol.



$f_0 = 656$ Hz

v = 324 m/s

Frequency heard due to movement of (S_1)

$$f_1 = \left(\frac{v}{v-u_s}\right) f_0$$

$$f_1 = \frac{324}{320} \times 656$$

And frequency heard due to movement of (S_2)

f_2 = 656 Hz

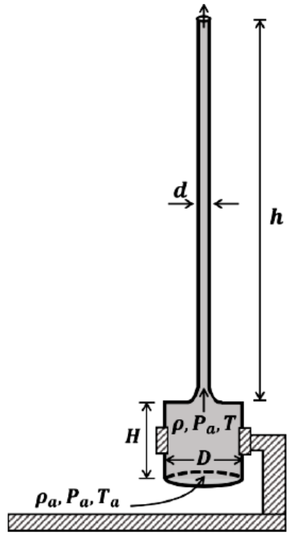
$\therefore$ Beat frequency $\Delta f = f_1 - f_2 = 656\left(\frac{324}{320} - 1\right)$

$\boxed{\Delta f = 8.2}$

PARAGRAPH II (Q.16 & 17)

A cylindrical furnace has height (H) and diameter (D) both 1m. It is maintained at temperature 360 K. The air gets heated inside the furnace at constant pressure P_a and its temperature becomes T = 360 K. The hot air with density ρ rises up a vertical chimney of diameter d = 0.1 m and height h = 9 m above the furnace and exits the chimney (see the figure). As a result, atmospheric air of density $\rho_a = 1.2$ Kg m^{-3}, pressure P_a and temperature T_a = 300K enters the furnace. Assume air as an ideal gas, neglect the variations in ρ and T inside the chimney and the furnace. Also ignore the viscous effects.

[Given: The acceleration due to gravity g = 10 ms^{-2} and π = 3.14]



Q.16 Considering the air flow to be streamline, the steady mass flow rate of air exiting the chimney is ____gm s^{-1}

Ans. **[47.10]**

Sol. $\because$ PM = ρRT

And P inside furnace is constant

$\therefore$ ρRT = constant

or ρT = constant

$\therefore \rho_a T_a = \rho T$

1.2(300) = ρ(360)

$\rho = 1$ kg/m^3

Now, applying Bernoulli's theorem at the bottom and the top of chimney

$$P_a + \frac{1}{2}\rho(0)^2 + 0 = (P_a - \rho_a gh) + \frac{1}{2}\rho(v^2) + \rho gh$$

$$v = \sqrt{\frac{2(\rho_a - \rho)gh}{\rho}}$$

$\Rightarrow v = 6$ m/s

$$\therefore \frac{dm}{dt} \text{ at exit} = \rho v\left(\frac{\pi d^2}{4}\right)$$

$$= \frac{1 \times 6 \times 3.14 \times (0.1)^2}{4}$$

= 0.0471 kg/s

= 47.10 g/s

Q.17 When the chimney is closed using a cap at the top, a pressure difference ΔP develops between the top and the bottom surfaces of the cap. If the changes in the temperature and density of the hot air, due to the stoppage of air flow, are negligible then the value of ΔP is ______Nm^{-2}

Ans. **[18.00]**

Sol.

CHEMISTRY

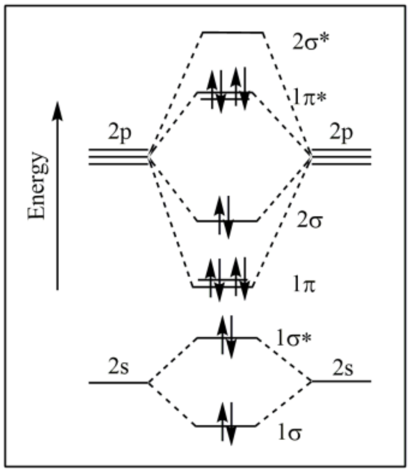
SECTION – 1 (Maximum Mark : 12)

- This section contains **FOUR (04)** questions.
- Each question has four options (A), (B), (C) and (D). **Only one** of these four option is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

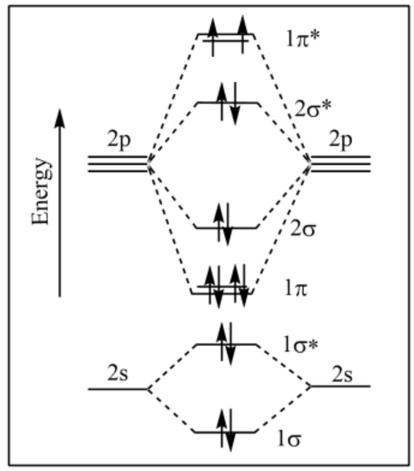
Full Marks	: **+3**	If **ONLY** the correct option is chosen;
Zero Marks	: **0**	If none of the option is chosen (i.e. the question is unanswered);
Negative Marks	: –1	In all other cases.

Q.1 The correct molecular orbital diagram for F_2 molecule in the ground state is

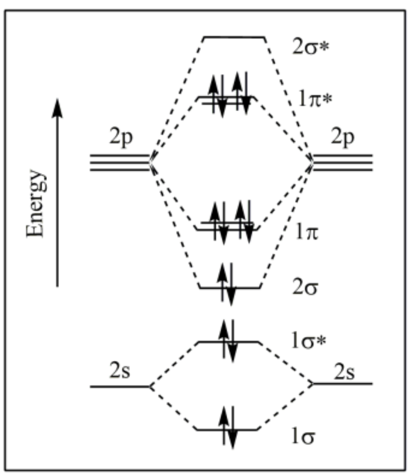
(A)



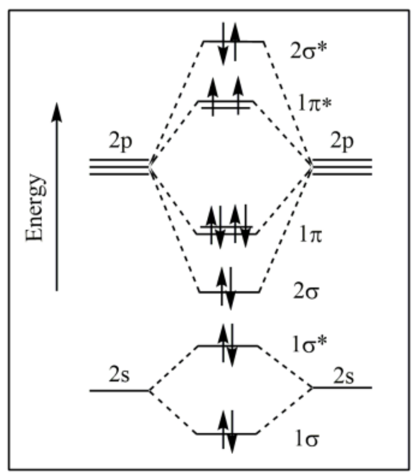
(B)



(C)

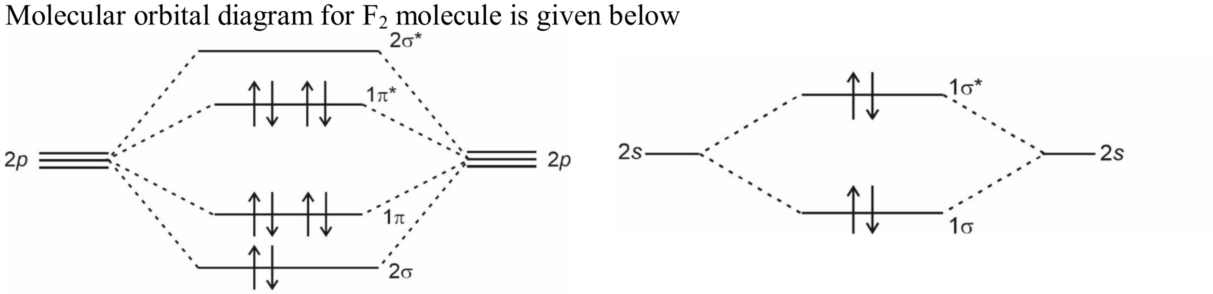


(D)



Ans. **[C]**

Sol. Molecular orbital diagram for F_2 molecule is given below



Hence the correct option is (C).

Q.2 Consider the following statements related to colloids.

(I) Lyophobic colloids are **not** formed by simple mixing of dispersed phase and dispersion medium.

(II) For emulsions, both the dispersed phase and the dispersion medium are liquid.

(III) Micelles are produced by dissolving a surfactant in any solvent at any temperature.

(IV) Tyndall effect can be observed from a colloidal solution with dispersed phase having the same refractive index as that of the dispersion medium.

The option with the correct set of statements is

(A) (I) and (II) (B) (II) and (III) (C) (III) and (IV) (D) (II) and (IV)

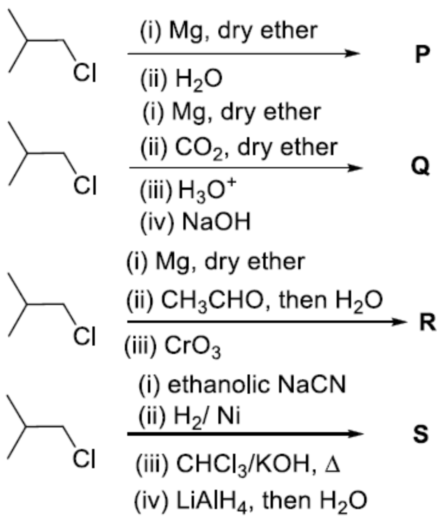
Ans. **[A]**

Sol. Lyophobic colloids are not formed by simple mixing of dispersed phase and dispersion medium. Their colloidal sols can be prepared only by special methods. Emulsion are colloids of liquid dispersed phase and liquid dispersion medium.

Micelles formation occurs when temperature is above a particular temperature called Kraft temperature T_K and concentration above a particular value know as critical micelle concentration (CMC).

Tyndall effect can be observed when the refractive indices of the dispersed phase and the dispersion medium differ greatly in magnitude.

Q.3 I n the following reactions, P, Q, R, and S are the major products.



The correct statement about **P**, **Q**, **R**, and **S** is

(A) **P** is a primary alcohol with four carbons.

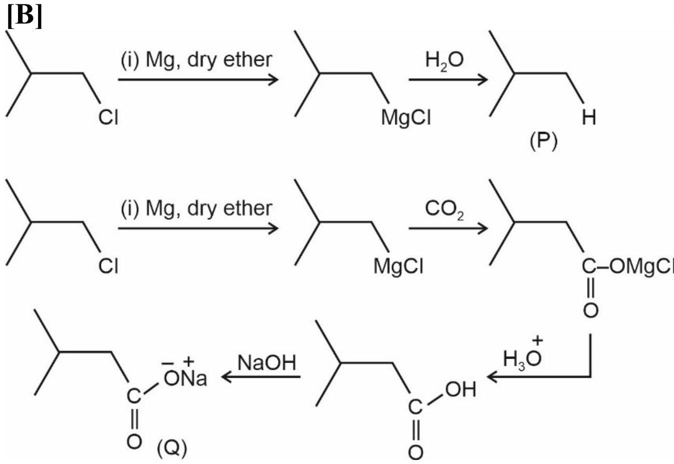
(B) **Q** undergoes Kolbe's electrolysis to give an eight-carbon product.

(C) **R** has six carbons and it undergoes Cannizzaro reaction.

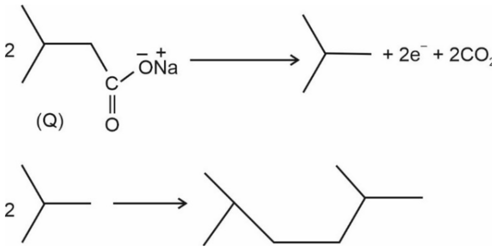
(D) **S** is a primary amine with six carbons.

Ans. **[B]**

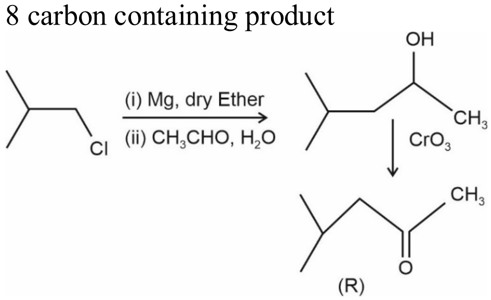
Sol.



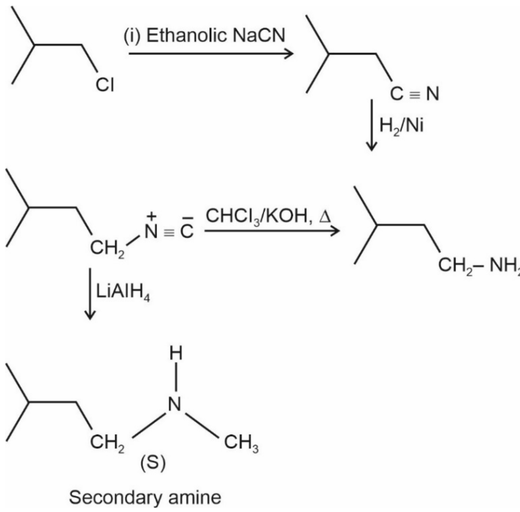
Kolbe electrolysis :



8 carbon containing product

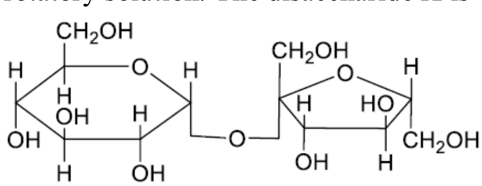


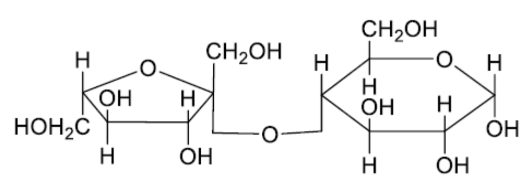
It undergoes aldol reaction not Cannizzaro.

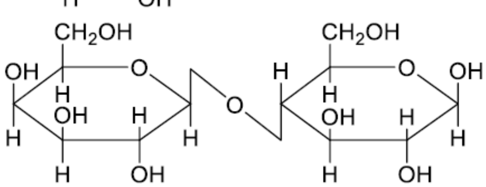


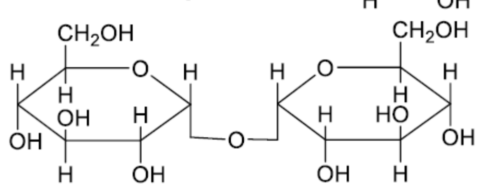
Since only Q fulfilled all the given conditions.
Hence, the correct option is (B).

Q.4 A disaccharide X cannot be oxidised by bromine water. The acid hydrolysis of **X** leads to a laevorotatory solution. The disaccharide **X** is

(A) 

(B) 

(C) 

(D) 

Ans. **[A]**

Sol. A and D cannot be oxidised by bromine water as they do not have hemiacetal linkage.
The acid hydrolysis of A leads to a laevorotatory solution.

A is sucrose which is dextrorotatory, on acid hydrolysis gives mixture of α-D-glucose and β-D-fructose, the mixture is laevorotatory.

SECTION – 2 (Maximum Mark : 12)

- This section contains **THREE (03)** questions
- Each question has **FOUR** options (A), (B), (C) and (D). **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	: **+4**	**ONLY** If (all) the correct option(s) is (are) chosen.
Partial Marks	: **+3**	If all the four options are correct but **ONLY** three options are chosen.
Partial Marks	: **+2**	If three or more options are correct but **ONLY** two options are chosen, both of which are correct.
Partial Marks	: **+1**	If two or more options are correct but **ONLY** one option is chosen and it is a correct option.
Zero Marks	: **0**	If unanswered;
Negative Marks	: **–2**	In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, than
choosing ONLY (A), (B) and (D) will get +4 marks;
choosing ONLY (A) and (B) will get +2 marks;
choosing ONLY (A) and (D) will get +2 marks;
choosing ONLY (B) and (D) will get +2 marks;
choosing ONLY (A) will get +1 mark;
choosing ONLY (B) will get +1 mark;
choosing ONLY (D) will get +1 mark;
choosing no option(s) (i.e. the question is unanswered) will get 0 marks and choosing any other option(s) will get –2 marks.

Q.5 The complex(es), which can exhibit the type of isomerism shown by $[Pt(NH_3)_2Br_2]$, is(are) [en = $H_2NCH_2CH_2NH_2$]

(A) $[Pt(en)(SCN)_2]$ (B) $[Zn(NH_3)_2Cl_2]$ (C) $[Pt(NH_3)_2Cl_4]$ (D) $[Cr(en)_2(H_2O)(SO_4)]^+$

Ans. **[C, D]**

Sol. $[Pt(NH_3)_2Cl_2]$ can exhibit G.I.

G.I. can be shown by

$[Pt(NH_3)_2Cl_4]$, $[Cr(en)_2(H_2O)(SO_4)]^+$

Q.6 Atoms of metals x, y, and z form face-centred cubic (fcc) unit cell of edge length L_x, body-centred cubic (bcc) unit cell of edge length L_y, and simple cubic unit cell of edge length L_z, respectively.

If $rz = \frac{\sqrt{3}}{2} r_y$; $r_y = \frac{8}{\sqrt{3}} r_x$; $M_z = \frac{3}{2} M_y$ and $M_z = 3M_x$, then the correct statement(s) is are

[Given: M_x, M_y, and M_z are molar masses of metals x, y, and z, respectively. r_x, r_y, and r_z are atomic radii of metals x, y, and z, respectively.]

(A) Packing efficiency of unit cell of x > Packing efficiency of unit cell of y > Packing efficiency of unit cell of z

(B) $L_y > L_z$

(C) $L_x > L_y$

(D) Density of x > Density of y

Ans. **[A,B,D]**

Sol. Metal x forms FCC (edge length L_x)

Metal y forms BCC (edge length L_y)

Metal z forms SC (edge length L_z)

Given $r_z = \frac{\sqrt{3}}{2} r_y$ and $r_y = \frac{8}{\sqrt{3}} r_x$

$\therefore\ r_z = \frac{\sqrt{3}}{2} \times \frac{8}{\sqrt{3}} r_x = 4r_x$

$M_z = \frac{3}{2} M_y$ $\quad M_x = 3M_x$

$\therefore\ M_y = 2M_x$

Packing efficiency FCC > BCC > SC

Packing efficiency unit cell x > y > z

In FCC unit cell:- atoms along the face diagonals are in contact.

$\therefore \sqrt{2}\, L_x = 4r_x \Rightarrow L_x = 2\sqrt{2}\ r_x$

In BCC unit cell: atoms along the body diagonal are

$\therefore\ \sqrt{3}\, L_y = 4r_y \Rightarrow L_y = \frac{4}{\sqrt{3}} r_y = \frac{4}{\sqrt{3}} \times \frac{8}{\sqrt{3}} r_x = \frac{32}{3} r_x$

$L_y = \frac{32}{3} r_x$

In SC unit cell, atoms along the edge are in contact

$\therefore L_z = 2r_z = 2 \times 4r_x = 8r_x$

$L_x = 2\sqrt{2}\, r_x$

$L_y = \frac{32}{3} r_x$

$L_z = 8r_x$

$\therefore\ L_y > L_z > L_x$

Density of x (Number of atoms of x per unit cell (z) = 4)

$$d_x = \frac{zM_x}{(L_x)^3 N_A} = \frac{4 \times M_x}{(2\sqrt{2} r_x)^3 \times N_A}$$

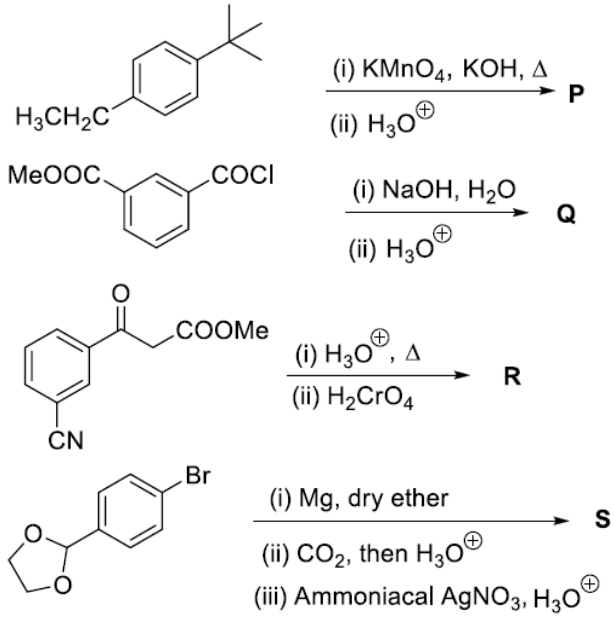
$$= \frac{4M_x}{16\sqrt{2} r_x^3 N_A} = \frac{M_x}{4\sqrt{2} r_x^3 N_A}$$

Density of y: (Number of atoms of y per unit cell (z) = 2)

$$d_y = \frac{zM_y}{(L_y)^3 N_A} = \frac{2 \times 2M_x}{\left(\frac{32}{3} r_x\right)^3 N_A} = \frac{108M_x}{32768 r_x^3 N_A}$$

$\therefore$ Density of x > density of y.

Q.7 In the following reactions, **P**, **Q**, **R**, and **S** are the major products.



The correct statement(s) about **P**, **Q**, **R**, and **S** is(are)

(A) **P** and **Q** are monomers of polymers dacron and glyptal, respectively.

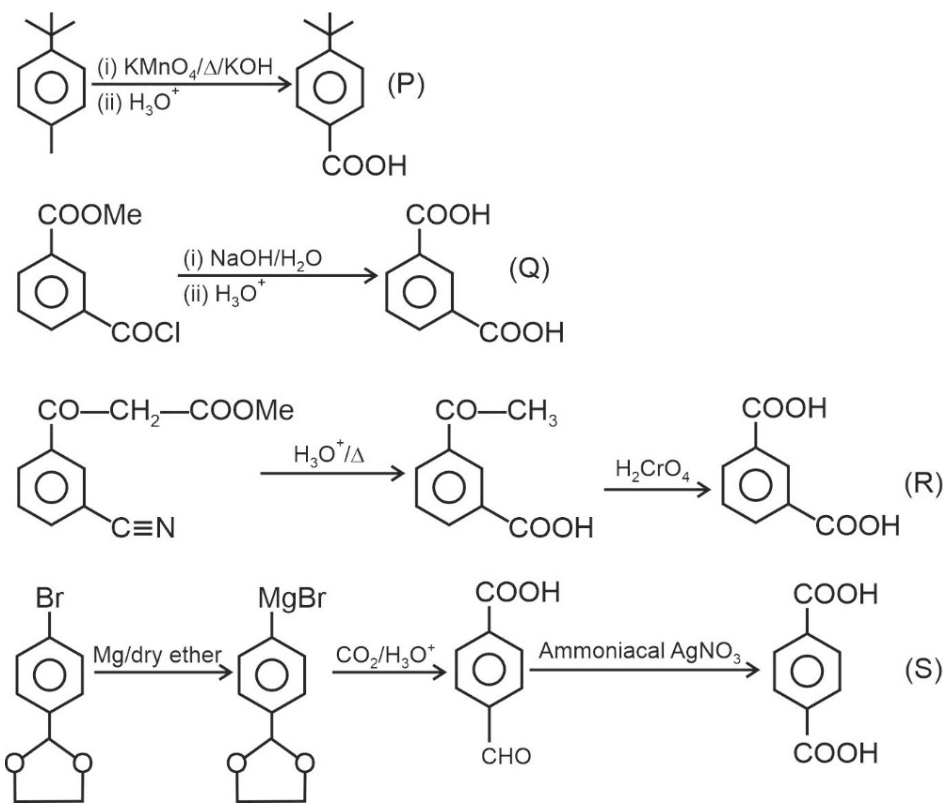
(B) **P**, **Q**, and **R** are dicarboxylic acids.

(C) Compounds **Q** and **R** are the same.

(D) **R** does **not** undergo aldol condensation and **S** does **not** undergo Cannizzaro reaction.

Ans. **[C, D]**

Sol.



SECTION – 3 (Maximum Marks: 24)

- This section contains **SIX (06)** questions.
- The answer to each question is a **NON-NEGATIVE INTERGER.**
- Four each question, enter the correct integer corresponding to the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	: **+4**	If **ONLY** the correct integer is entered;
Zero Marks	: **0**	In all other cases.

Q.8 H_2S (5 moles) reacts completely with acidified aqueous potassium permanganate solution. In this reaction, the number of moles of water produced is **x**, and the number of moles of electrons involved is **y**. The value of (**x** + **y**) is ____.

Ans. [18]

Sol. $KMnO_4 \xrightarrow{H^+} Mn^{+2}$

$S^{-2} \longrightarrow \overset{0}{S}$

$\therefore$ n_{factor} of $KMnO_4 = 5$

n_{factor} of $S^{-2}(H_2S) = 2$

$(n_{KMnO_4} \times 5) = (5 \times 2)_{H_2S}$

$[(GEN)_{KMnO_4} = (GEP)_{H_2S}]$

$\therefore n_{KMnO_4} = 2$

$\therefore$ 2 $KMnO_4 + 3H_2SO_4 + 5H_2S \rightarrow K_2SO_4 + 2MnSO_4 + 5S + 8H_2O$

Number of moles of water produced = '8'

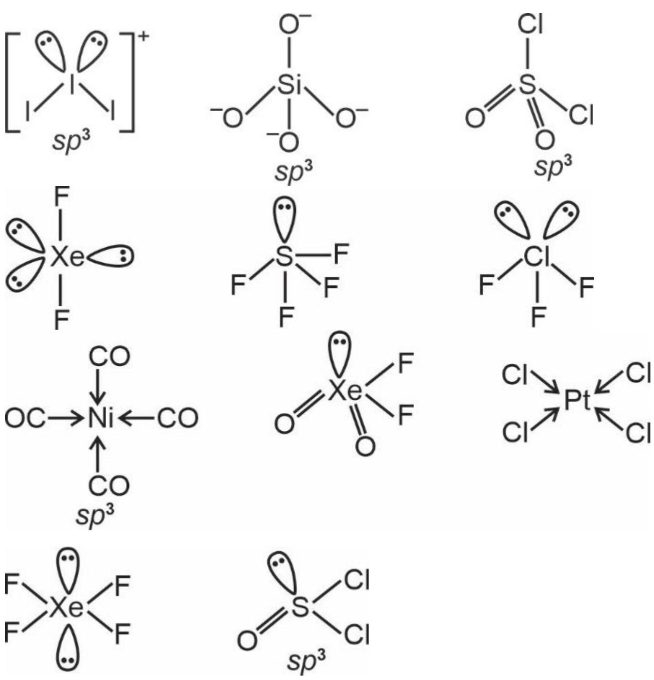
Number of moles of electrons involved = 10

$\therefore$ x = 8, y = 10 $\Rightarrow$ (x + y) = 18

Q.9 A mong $[I_3]^+$, $[SiO_4]^{4-}$, SO_2Cl_2, XeF_2, SF_4, ClF_3, $Ni(CO)_4$, XeO_2F_2, $[PtCl_4]^{2-}$, XeF_4, and $SOCl_2$, the total number of species having sp^3 hybridised central atom is ______.

Ans. **[5]**

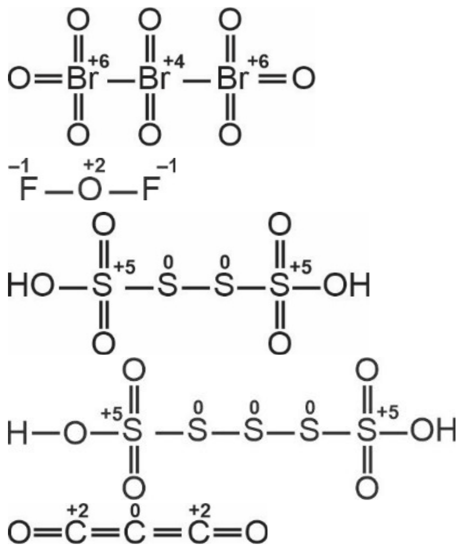
Sol.



Q.10 Consider the following molecules: Br_3O_8, F_2O, $H_2S_4O_6$, $H_2S_5O_6$, and C_3O_2.
Count the number of atoms existing in their zero oxidation state in each molecule.
Their sum is____.

Ans. **[6]**

Sol.



Total atom with zero oxidation number state are 6.

Q.11 For He^+, a transition takes place from the orbit of radius 105.8 pm to the orbit of radius 26.45 pm. The wavelength (in nm) of the emitted photon during the transition is ___.
[**Use**: Bohr radius, a = 52.9 pm; Rydberg constant, $R_H = 2.2 \times 10^{-18}$ J; Planck's constant, $h = 6.6 \times 10^{-34}$ J s; Speed of light, $c = 3 \times 10^8$ m s^{-1}]

Ans. **[30]**

Sol. $r = 52.9 \times \frac{n^2}{z}$ pm

$\therefore \quad 105.8 = \frac{52.9 \times n^2}{2} \qquad \therefore \quad n_2 = 2$

and $\quad 26.45 = 52.9 \times \frac{n^2}{2} \qquad \therefore \quad n_1 = 1$

$$\because \quad \Delta E = R_H hC \times z^2 \left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right]$$

$$\frac{hc}{\lambda} = R_H hC \times z^2 \left[\frac{1}{n_1^2} - \frac{1}{n_2^2}\right]$$

$$\frac{6.6 \times 10^{-34} \times 3 \times 10^8}{\lambda} = 2.2 \times 10^{-18} \times 4 \times \left[\frac{1}{1} - \frac{1}{4}\right]$$

$$\frac{6.6 \times 10^{-34} \times 3 \times 10^8}{\lambda} = 2.2 \times 10^{-18}\ 4 \times \frac{3}{4}$$

$\therefore \quad \lambda = 300$ Å

$\therefore \quad \lambda = 30$ nm

Q.12 50 mL of 0.2 molal urea solution (density = 1.012 g mL^{-1} at 300 K) is mixed with 250 mL of a solution containing 0.06 g of urea. Both the solutions were prepared in the same solvent. The osmotic pressure (in Torr) of the resulting solution at 300 K is ___.
[**Use**: Molar mass of urea = 60 g mol^{-1}; gas constant, R = 62 L Torr K^{-1} mol^{-1}; Assume, $\Delta_{mix}H = 0$, $\Delta_{mix}V = 0$]

Ans. **[682]**

Sol. Mole of urea = 0.2
Weight of urea = 0.2 × 60 = 12 g
Weight of solvent = 1000 g
Weight of solution = 1012 g

$\therefore$ Volume of solution = $\frac{1012}{1.012}$ = 1000 ml

$\because$ 1000 ml solution contain 0.2 mole

$\therefore$ 50 ml solution contain = $\frac{0.2 \times 50}{1000}$ = 0.01

Mole of urea in other solution = $\frac{0.06}{60}$ = 0.001

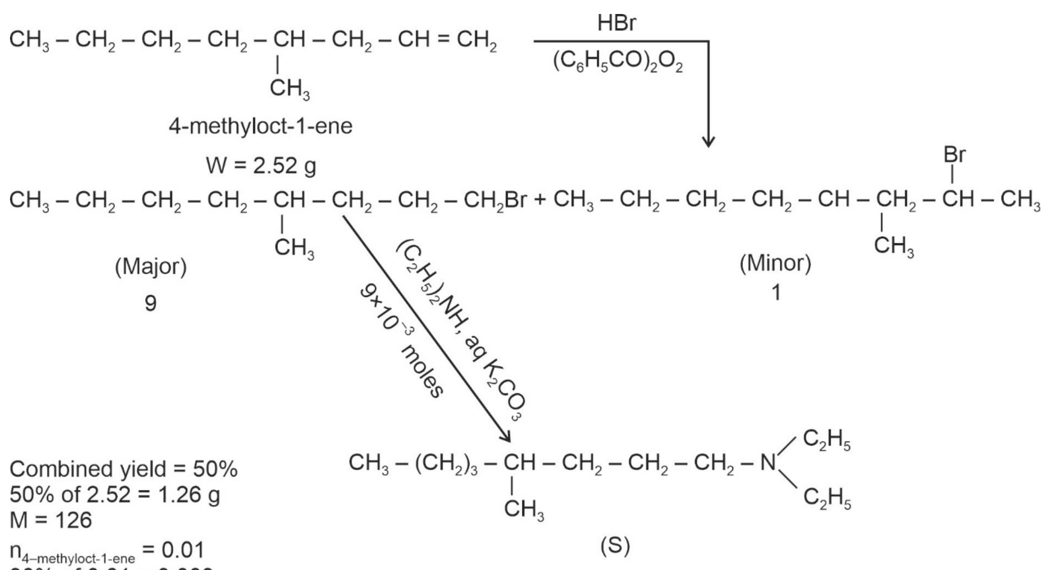
$\therefore$ Concentration of solution = $\frac{0.01 + 0.001}{\frac{300}{1000}}$ = 0.0366

$\therefore$ π = CRT
= 0.0366 × 62 × 300= 682

Q.13 The reaction of 4-methyloct-1-ene (**P**, 2.52 g) with HBr in the presence of $(C_6H_5CO)_2O_2$ gives two isomeric bromides in a 9 : 1 ratio, with a combined yield of 50%. Of these, the entire amount of the primary alkyl bromide was reacted with an appropriate amount of diethylamine followed by treatment with aq. K_2CO_3 to give a non-ionic product **S** in 100% yield.
The mass (in mg) of S obtained is ___.
[**Use** molar mass (in g mol^{-1}): H = 1, C = 12, N = 14, Br = 80]

Ans. **[1791]**

Sol.



Combined yield = 50%
50% of 2.52 = 1.26 g
M = 126
$n_{4\text{-methyloct-1-ene}}$ = 0.01
90% of 0.01 = 0.009

Mass of S = 0.009 × 199
= 1.791 g
= 1791 mg

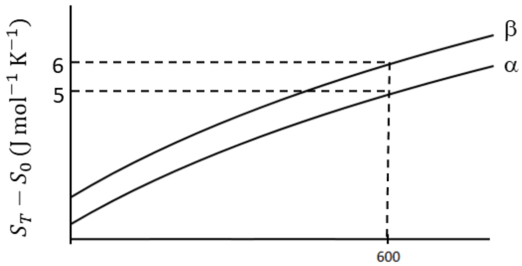
SECTION – 4 (Maximum Marks: 12)

- This section contains **TWO (02)** paragraphs.
- Based on each paragraph, there are **TWO (02)** questions.
- The answer to each question is a **NUMERICAL VALUE.**
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer.
- If the numerical value has more than two decimal places, **truncate/round-off** the value to **TWO** decimal places.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks : **+3** f **ONLY** the correct numerical value is entered in the designated place;
Zero Marks : **0** In all other cases.

"PARAGRAPH I" (Q.14 & 15)

The entropy versus temperature plot for phases α and β at 1 bar pressure is given.
S_T and S_0 are entropies of the phases at temperatures T and 0 K, respectively.



The transition temperature for α to β phase change is 600 K and $C_{p,\beta} - C_{p,\alpha}$ = 1 J mol^{-1} K^{-1}. Assume ($C_{p,\beta} - C_{p,\alpha}$) is independent of temperature in the range of 200 to 700 K. $C_{p,\alpha}$ and $C_{p,\beta}$ are heat capacities of α and β phases, respectively.

Q.14 The value of entropy change, $S_\beta - S_\alpha$ (in J mol^{-1} K^{-1}), at 300 K is ___.
[Use: ln 2 = 0.69; Given: $S_\beta - S_\alpha = 0$ at 0 K]

Ans. **[0.31]**

Sol. $\Delta S_{600} - \Delta S_{300} = \int_{300}^{600} \frac{1\times(C_{p.\beta} - C_{p.\alpha})dT}{T}$

$= 1 \times 1 \times \left(\ln\frac{T_2}{T_1}\right) \begin{pmatrix} T_2 = 600k \\ T_1 = 300K \end{pmatrix}$

$1 - \Delta S_{300} = 1 \times 1 \times \ln 2$

$\Delta S_{300} = 1 - 0.69$

$\Delta S_{300} = 0.31$ J mol^{-1} K^{-1}

Q.15 The value of enthalpy change, $H_\beta - H_\alpha$ (in J mol^{-1}), at 300 K is ___.

Ans. **[300]**

Sol. $\Delta H_{600} - \Delta H_{300} = 1 \times (C_{p,\beta} - C_{p,\alpha})\,(600 - 300)$

Now, at transition temperature,

$\Delta H_{600} = T\Delta S_{600}$

$= 600 \times (6 - 5)$

$= 600 \text{ J mol}^{-1}$

$600 - \Delta H_{300} = 1 \times 1 \times 300$

$\Delta H_{300} = 600 - 300$

$= 300 \text{ J mol}^{-1}$

"PARAGRAPH II" (Q.16 & 17)

A trinitro compound, 1,3,5-tris-(4-nitrophenyl)benzene, on complete reaction with an excess of Sn/HCl gives a major product, which on treatment with an excess of $NaNO_2$/HCl at 0°C provides **P** as the product.

P, upon treatment with excess of H_2O at room temperature, gives the product **Q**. Bromination of **Q** in aqueous medium furnishes the product **R**. The compound **P** upon treatment with an excess of phenol under basic conditions gives the product **S**.

The molar mass difference between compounds **Q** and **R** is 474 g mol–1 and between compounds **P** and **S** is 172.5 g mol^{-1}.

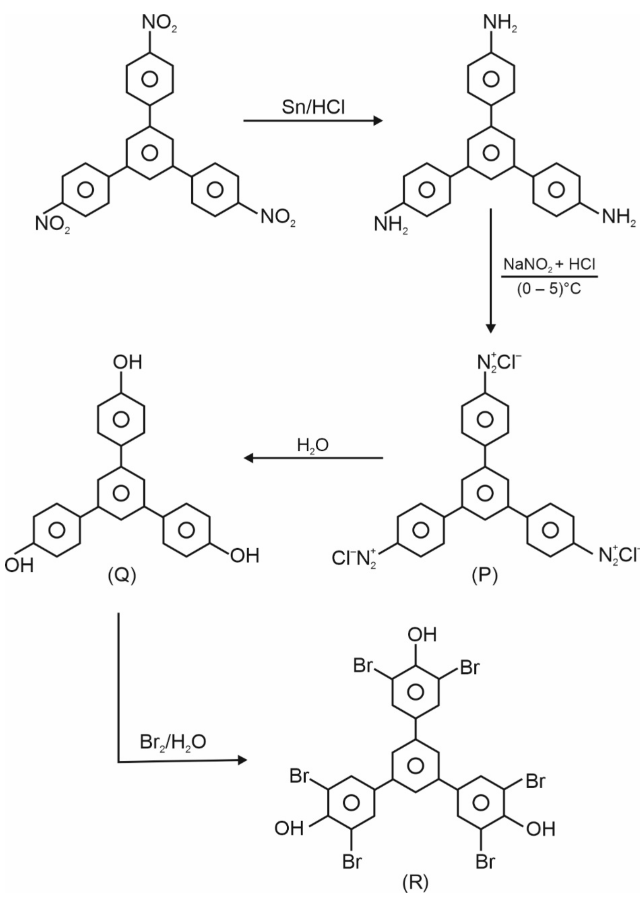
Q.16 The number of heteroatoms present in one molecule of **R** is ______ .

[**Use**: Molar mass (in g mol^{-1}): H = 1, C = 12, N = 14, O = 16, Br = 80, Cl = 35.5

Atoms other than C and H are considered as heteroatoms]

Ans. **[9]**

Sol.

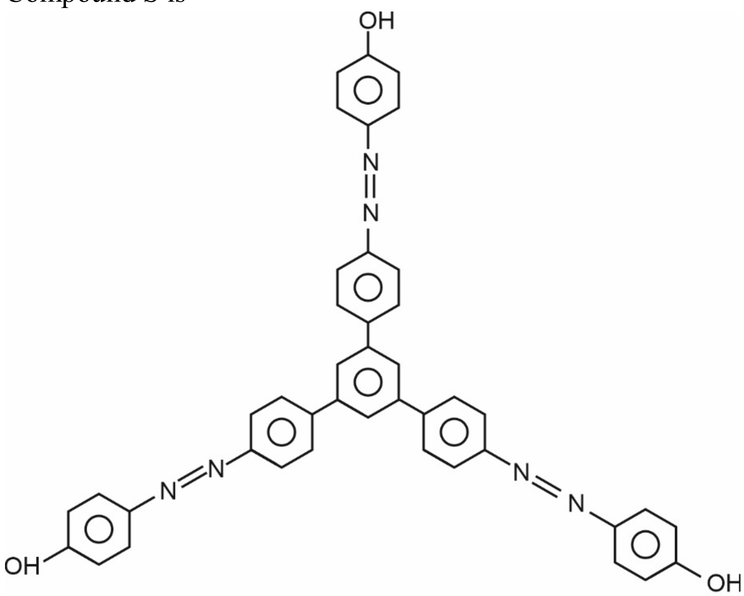


Number of Heteroatoms in R is 9.

Q.17 The total number of carbon atoms and heteroatoms present in one molecule of **S** is ______ .
[**Use**: Molar mass (in g mol–1): H = 1, C = 12, N = 14, O = 16, Br = 80, Cl = 35.5
Atoms other than C and H are considered as heteroatoms]

Ans. **[51]**

Sol. Compound S is



(S)

Number of carbon atoms + Number of Heteroatoms = 51