CAREER POINT

JEE Advanced Exam 2019

(Paper & Solution)

Date : 27 / 05 / 2019

PAPER-1

PART-I (PHYSICS)

SECTION – 1 (Maximum Marks : 12)

- This section contains **FOUR (04)** questions
- Each question has **FOUR** options. **ONLY ONE** of these four options is correct answer.
- For each question, choose the correct option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	: **+3**	If ONLY the correct option is chosen.
Zero Marks	: **0**	If none of the options is chosen (i.e. the question is unanswered).
Negative Marks	: **–1**	In all other cases.

Q.1 Consider a spherical gaseous cloud of mass density ρ(r) in free space where r is the radial distance from its center. The gaseous cloud is made of particles of equal mass m moving in circular orbits about the common center with the same kinetic energy K. The force acting on the particle is their mutual gravitational force. If ρ(r) is constant in time, the particle number density n(r) = ρ(r)/m is

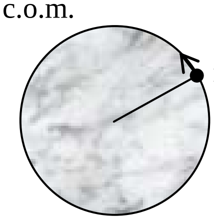
[G is universal gravitational constant]

(1) $\frac{3K}{\pi r^2 m^2 G}$ (2) $\frac{K}{2\pi r^2 m^2 G}$ (3) $\frac{K}{\pi r^2 m^2 G}$ (4) $\frac{K}{6\pi r^2 m^2 G}$

Ans.[2]

Sol. Let the mass of cloud = M

Consider a particle at distance r from c.o.m.



$\Rightarrow \quad \frac{GMm}{r^2} = \frac{mv^2}{r}$ ……(1)

Given that $\frac{1}{2}mv^2 = K$

$v^2 = \frac{2K}{m}$

Put the value in equ.(1)

$\frac{GM}{r} = v^2 = \frac{2K}{m}$

$M = \frac{2Kr}{Gm}$

Differentiating it

$\frac{dM}{dr} = \frac{2K}{Gm}$

Put dM = mass of element = $(4\pi r^2 dr)\rho$

$4\pi r^2\rho = \frac{2K}{Gm}$

$\therefore\ \rho = \frac{2K}{Gm(4\pi r^2)} = \frac{K}{2\pi r^2 Gm}$

$\frac{\rho}{m} = \frac{K}{2\pi r^2 Gm^2}$

Q.2 In a radioactive sample, ${}^{40}_{19}K$ nuclei either decay into stable ${}^{40}_{20}Ca$ nuclei with decay constant 4.5×10^{-10} per year or into stable ${}^{40}_{18}Ar$ nuclei with decay constant 0.5×10^{-10} per year. Given that in this sample all the stable ${}^{40}_{20}Ca$ and ${}^{40}_{18}Ar$ nuclei are produced by the ${}^{40}_{19}K$ nuclei only. In time $t \times 10^9$ years, if the ratio of the sum of stable ${}^{40}_{20}Ca$ and ${}^{40}_{18}Ar$ nuclei to the radioactive ${}^{40}_{19}K$ nuclei is 99, the value of t will be- [Given ln 10 = 2.3]

(1) 2.3 (2) 9.2 (3) 1.15 (4) 4.6

Ans.[2]

Sol. $\lambda = \lambda_1 + \lambda_2 = 4.5 \times 10^{-10} + 0.5 \times 10^{-10}$

$\lambda = 5.0 \times 10^{-10}$ per year

$N = N_0 e^{-\lambda t}$ ……..(1)

In time t 99% K decayed

Undecayed

$N = N_0 - \frac{99N_0}{100} = \frac{N_0}{100}$

Put the value in equation (1)

$\frac{N_0}{100} = N_0 e^{-\lambda t}$

$t = \frac{(2.303)\times 2}{\lambda}$

$\therefore t = \frac{4.6}{\lambda} = \frac{4.6}{5} \times 10^{10}$ years

$\frac{4.6\times 10^{10}}{5} = t \times 10^9$

$t = \frac{4.6\times 10^{10}}{5\times 10^9} = \frac{46}{5} = 9.2$

Q.3 A thin spherical insulating shell of radius R carries a uniformly distributed charge such that the potential at its surface is V_0. A hole with a small area $\alpha 4\pi R^2$ ($\alpha << 1$) is made on the shell without affecting the rest of the shell. Which one of the following statements is correct ?

(1) The magnitude of electric field at the center of the shell is reduced by $\frac{\alpha V_0}{2R}$

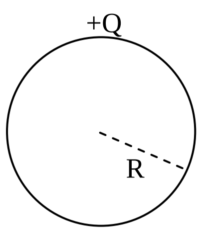
(2) The ratio of the potential at the center of the shell to that of the point at $\frac{1}{2}$R from center towards the hole will be $\frac{1-\alpha}{1-2\alpha}$

(3) The magnitude of electric field at a point, located on a line passing through the hole and shell's center, on a distance 2R from the center of the spherical shell will be reduced by $\frac{\alpha V_0}{2R}$

(4) The potential at the center of the shell is reduced by $2\alpha V_0$

Ans.[2]

Sol. Potential at surface = $V_0 = \frac{KQ}{R}$



Small element of area 'da' is removed charge on element $dq = \sigma da$

$$dq = \frac{Q}{4\pi R^2} da$$

$$= \frac{Q}{4\pi R^2}(4\pi R^2 \alpha)$$

$$dq = \alpha Q$$

Potential at center now

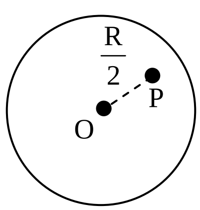
$$V' = \frac{kQ}{R} - \frac{k(dq)}{R} = V_0 (1 - \alpha)$$

Initially $V = V_0 = \frac{kQ}{R}$

Decrease in potential = $V_0 - V_0(1 - \alpha)$

$= V_0 \alpha$

Potential at distance $\frac{R}{2}$ from center



$$V_P = \frac{KQ}{R} - \frac{2Kdq}{R} = \frac{KQ}{R}(1-2\alpha) = V_0(1 - 2\alpha)$$

Therefore $\frac{V_C}{V_P} = \frac{1-\alpha}{1-2\alpha}$

Q.4 A current carrying wire heats a metal rod. The wire provides a constant power (P) to the rod. The metal rod is enclosed in an insulated container. It is observed that the temperature (T) in the metal rod changes with time (t) as $T(t) = T_0(1 + \beta t^{1/4})$, where β is a constant with appropriate dimension while T_0 is a constant with dimension of temperature. The heat capacity of the metal is-

(1) $\frac{4P(T(t)-T_0)^3}{\beta^4 T_0^4}$ (2) $\frac{4P(T(t)-T_0)}{\beta^4 T_0^2}$ (3) $\frac{4P(T(t)-T_0)^4}{\beta^4 T_0^5}$ (4) $\frac{4P(T(t)-T_0)^2}{\beta^4 T_0^3}$

Ans.[1]

Sol. Heat Q = ms dT

$\therefore$ Power $P = \frac{dQ}{dt} = \frac{msdT}{dt}$

$\therefore P = ms\frac{dT}{dt}$ … (1)

$T = T_0(1 + \beta\, t^{1/4})$

$\frac{dT}{dt} = \frac{\beta T_0}{4} t^{-3/4}$ put the value in (1)

$\therefore P = ms\left(\frac{\beta T_0}{4} t^{-3/4}\right)$

Heat capacity $= ms = \frac{4P}{\beta T_0 t^{-3/4}} = \frac{4P}{\beta T_0} t^{3/4}$ … (2)

$t^{1/4} = \frac{T - T_0}{T_0 \beta}$

$\therefore t^{3/4} = \frac{(T-T_0)^3}{T_0^3 \beta^3}$ put the value in (2)

Heat capacity $= \frac{4P}{\beta T_0} \frac{[T-T_0]^3}{T_0^3 \beta^3} = \frac{4P(T-T_0)^3}{\beta^4\ T_0^4}$

SECTION – 2 (Maximum Marks : 32)

- This section contains **EIGHT (08)** questions
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is (are) correct option(s).
- For each question, choose(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	: **+4**	If only (all) the correct option(s) is (are) chosen.
Partial Marks	: **+3**	If all the four options are correct but ONLY three options are chosen.
Partial Marks	: **+2**	If three or more options are correct but ONLY two options are chosen, both of which are correct options.
Partial Marks	: **+1**	If two or more options are correct but ONLY one option is chosen and it is a correct option.
Zero Marks	: **0**	If none of the option is chosen (i.e. the question is unanswered).
Negative Marks	: **–1**	In all other cases.

Q.1 A charged shell of a radius R carries a total charge Q. Given ϕ as the flux of electric field through a closed cylindrical surface of height h, radius r and with its center same as that of the shell. Here, center of the cylinder is a point on the axis of the cylinder which is equidistant from its top and bottom surfaces. Which of the following option(s) is/are correct ? [$\in_0$ is the permittivity of free space]

(1) If h > 2R and r > R then $\phi = Q/\in_0$

(2) If h < 8R/5 and r = 3R/5 then $\phi = 0$

(3) If h > 2R and r = 3R/5 then $\phi = Q/5\in_0$

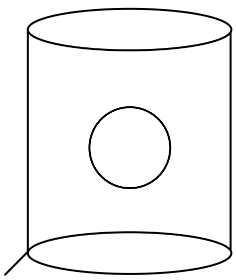
(4) If h > 2R and r = 4R/5 then $\phi = Q/5\in_0$

Ans.[1,2,3]

Sol. (1) h > 2R, r > R

$q_{en} = Q$

$\phi = \frac{Q}{\in_0}$

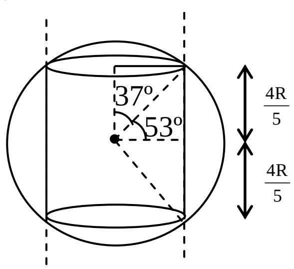


(2) $h < \frac{8R}{5}$, $r = \frac{3R}{5}$

h < 1.6 R

$2r = \frac{6}{5}R = 1.2\text{ R}$

$\phi = 0$



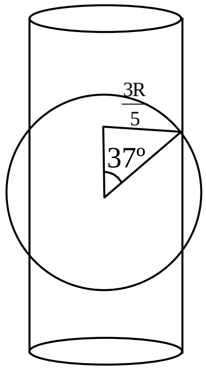
(3) h > 2R

$= 2\pi\left(1 - \frac{4}{5}\right)$

$= 2\pi\left(\frac{1}{5}\right) = \frac{2\pi}{5}$

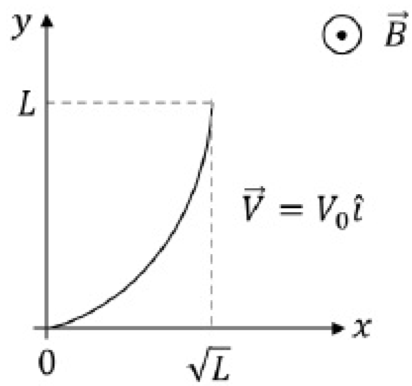
$q_{en} = 2\left[\frac{Q}{4\pi} \times \frac{2\pi}{5}\right]$

$= \frac{Q}{5} \Rightarrow \phi = \frac{Q}{5\in_0}$



option (1), (2), (3)

Q.2 A conducting wire of parabolic shape, initially $y = x^2$, is moving with velocity $\vec{V} = V_0\hat{i}$ in a non-uniform magnetic field $\vec{B} = B_0\left(1+\left(\frac{y}{L}\right)^{\beta}\right)\hat{k}$, as shown in figure. If V_0, B_0, L and β are positive constants and Δϕ is the potential difference developed between the ends of the wire, then the correct statement(s) is/are



(1) $|\Delta\phi| = \frac{1}{2}B_0V_0L$ for $\beta = 0$

(2) $|\Delta\phi|$ is proportional to the length of the wire projected on the y-axis

(3) $|\Delta\phi| = \frac{4}{3}B_0V_0L$ for $\beta = 2$

(4) $|\Delta\phi|$ remains the same if the parabolic wire is replaced by a straight wire, $y = x$ initially, of length $\sqrt{2}\,L$

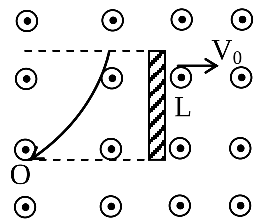
Ans.[2,3,4]

Sol. Magnetic field is uniform, therefore emf can be calculated for straight wire of length L

$$e = \int_0^L BV_0\, dy$$

$$e = B_0V_0\int\left(1+\left(\frac{y}{L}\right)\right)^{\beta} dy$$

$$= B_0V_0\left[y + \frac{1}{L^\beta}\frac{(y)^{\beta+1}}{\beta+1}\right]_0^L$$



$$= B_0V_0\left[L + \frac{L^{\beta+1}}{(\beta+1)L^\beta}\right]$$

$$= B_0V_0L\left[1+\frac{1}{\beta+1}\right]$$

$\therefore$ If $\beta = 0$, $e = B_0V_0L[2] = 2B_0V_0L$

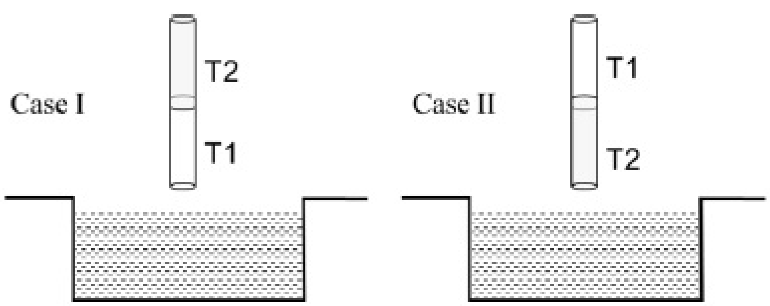
$e \propto L$ [2] is correct

If $\beta = 2$, $e = B_0V_0L\left[1+\frac{1}{3}\right] = \frac{4}{3}B_0V_0L$ [3] is correct

If x = y, still length of projection is L

$\therefore$ e same 4 is correct

Q.3 A cylindrical capillary tube of 0.2 mm radius is made by joining two capillaries T1 and T2 of different materials having water contact angles of 0º and 60º, respectively. The capillary tube is dipped vertically in water in two different configurations, case I and II as shown in figure. Which of the following option(s) is(are) correct ? [Surface tension of water = 0.075 N/m, density of water = 1000 kg/m^3, take g = 10 m/s^2]



(1) For case I, if the capillary joint is 5 cm above the water surface, the height of water column raised in the tube will be more than 8.75 cm. (Neglect the weight of the water in the meniscus)

(2) For case I, if the joint is kept at 8 cm above the water surface, the height of water column in the tube will be 7.5 cm. (Neglect the weight of the water in the meniscus)

(3) For case II, if the capillary joint is 5 cm above the water surface, the height of water column raised in the tube will be 3.75 cm. (Neglect the weight of the water in the meniscus)

(4) The correction in the height of water column raised in the tube, due to weight of water contained in the meniscus, will be different for both cases

Ans.[2,3,4]

Sol. **Case I**

$\frac{2T_1}{r} = \rho gh$

$h = \frac{2.(0.075)}{\rho gr}$

$= \frac{2.(0.075)}{10^3 \times 10 \times 0.2 \times 10^{-3}}$

$= 0.075$ m

$= 7.5$ cm

But capillary joint is at 5 cm from water surface.

$\Rightarrow T_1$ section of the capillary is completely filled.

Now 2.5 cm of the liquid in T_1 will take half the length in T_2

$\Rightarrow$ total length = 5 cm + $\frac{2.5}{2}$ = 6.25 cm

Case II

$\rho gh = \frac{2T}{r}\cos\theta$

$\Rightarrow h = 3.75$ cm

In option (4)

End correction will be different.

So option (2), (3) and (4) are correct.

Q.4 Let us consider a system of units in which mass and angular momentum are dimensionless. If length has dimension of L, which of the following statement(s) is/are correct ?

(1) The dimension of power is L^{-5}

(2) The dimension of linear momentum is L^{-1}

(3) The dimension of energy is L^{-2}

(4) The dimension of force is L^{-3}

Ans.[2,3,4]

Sol. $m \to 0$, $v = \frac{1}{L} = L^{-1}$

$mvr \to 0$ $\qquad M \to 0$

$L \to L$ $\qquad L \to L'$

$\frac{L}{T} \to L'$

$T = L^2$

option(1)

Power $P = M^1L^2T^{-3}$

$= M^0L^2(L^2)^{-3} = L^{-4}$ incorrect

option(2)

Momentum = mv = M^0L^{-1} correct

option(3)

Energy = $M^1L^2T^{-2}$

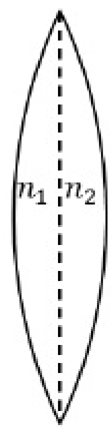
$= M^0L^2L^{-4} \rightarrow L^{-2}$ correct

option (4)

F = $M^1L^1T^{-2}$

$= M^0L^1L^{-4} = L^{-3}$ correct

Q.5 A thin convex lens is made of two materials with refractive indices n_1 and n_2, as shown in the figure. The radius of curvature of the left and right spherical surfaces are equal. f is the focal length of the lens when $n_1 = n_2 = n$. The focal length is $f + \Delta f$ when $n_1 = n$ and $n_2 = n + \Delta n$. Assuming $\Delta n << (n - 1)$ and $1 < n < 2$, the correct statement(s) is/are



(1) If $\frac{\Delta n}{n} < 0$ then $\frac{\Delta f}{f} > 0$

(2) For n = 1.5, $\Delta n = 10^{-3}$ and f = 20 cm, the value of $|\Delta f|$ will be 0.02 cm (round off to 2nd decimal place)

(3) The relation between $\frac{\Delta f}{f}$ and $\frac{\Delta n}{n}$ remains unchanged if both the convex surfaces are replaced by concave surfaces of the same radius of curvature

(4) $\left|\frac{\Delta f}{f}\right| < \left|\frac{\Delta n}{n}\right|$

Ans.[1,2,3]

Sol. $\frac{1}{f} = (n-1)\left(\frac{1}{R}\right)$, $\frac{1}{f_0} = \frac{1}{f} + \frac{1}{f} = \frac{2}{f}$

$\frac{1}{f_2} = (n + \Delta n - 1)\frac{1}{R}$

$\frac{1}{f + \Delta f} = \frac{(n-1)}{R} + (n + \Delta n - 1)\frac{1}{R}$

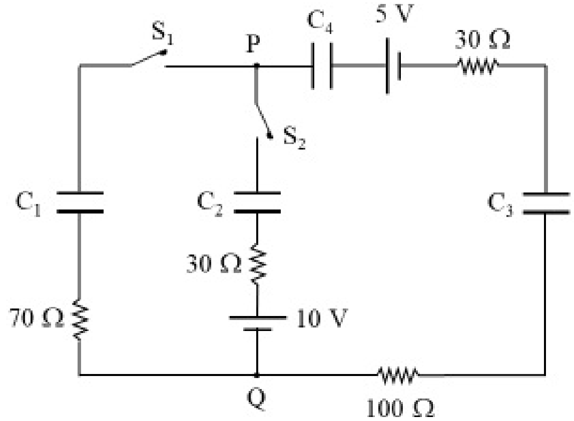
$$\frac{1}{f+\Delta f} = \frac{2n+\Delta n-2}{R}$$

$$\frac{f_0+\Delta f_0}{f_0} = \frac{\frac{2(n-1)}{R}}{\frac{2n+\Delta n-2}{R}} \Rightarrow 1 + \frac{\Delta f_0}{f_0} = \frac{2(n-1)}{2n+\Delta n-2}$$

$$\frac{\Delta f_0}{f_0} = \frac{-\Delta n}{2n+\Delta n-2} \Rightarrow \Delta f_0 = (20)\left[\frac{10^{-3}}{3+10^{-3}-2}\right] = 0.02 \text{ cm}$$

Option (1), (2), (3)

Q.6 In the circuit shown, initially there is no charge on capacitors and keys S_1 and S_2 are open. The values of the capacitors are C_1 = 10 μF, C_2 = 30 μF and $C_3 = C_4$ = 80 μF.



Which of the statement(s) is/are correct ?

(1) At time t = 0, the key S_1 is closed, the instantaneous current in the closed circuit will be 25 mA

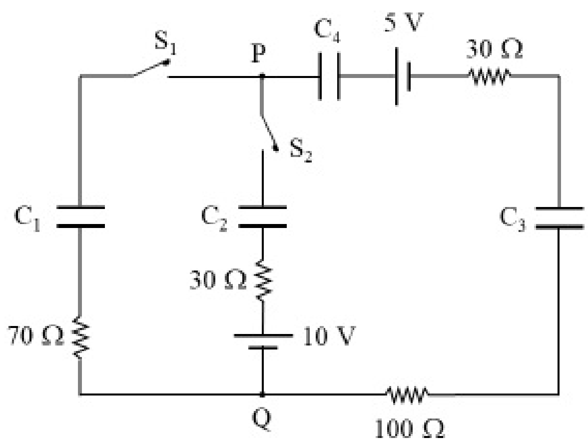
(2) The key S_1 is kept closed for long time such that capacitors are fully charged. Now key S_2 is closed, at this time, the instantaneous current across 30 Ω resistor (between points P and Q) will be 0.2 A (round off to 1st decimal place)

(3) If key S_1 is kept closed for long time such that capacitors are fully charged, the voltage across the capacitor C_1 will be 4 V

(4) If key S_1 is kept closed for long time such that capacitors are fully charged, the voltage difference between points P and Q will be 10 V

Ans.[1,3]

Sol.



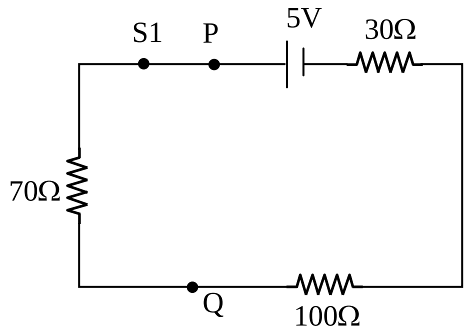
At t = 0, capacitors will be short circuited

$R_{eq} = 200\ \Omega$

$i = \frac{5}{200} = \frac{1}{40}$

$= 0.025$ A

$= 25$ mA



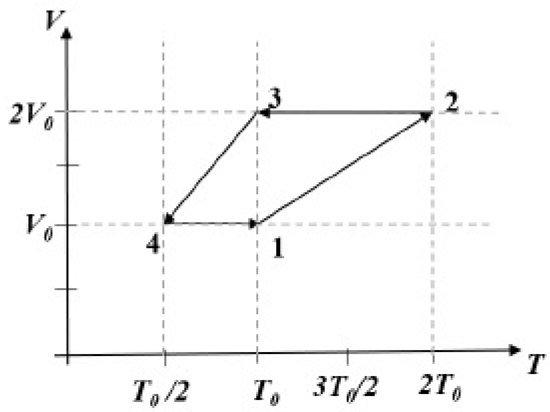
∴ (1) is correct

After long time

potential at $C_1 \Rightarrow V_1 = \frac{40 \times 5}{50} = 4V$ (3) is correct

So (1) and (3) are correct

Q.7 One mole of a monatomic ideal gas goes through a thermodynamic cycle, as shown in the volume versus temperature (V-T) diagram. The correct statement(s) is/are [R is the gas constant]



(1) The above thermodynamic cycle exhibits only isochoric and adiabatic processes.

(2) Work done in this thermodynamic cycle $(1 \to 2 \to 3 \to 4 \to 1)$ is $|W| = \frac{1}{2}RT_0$

(3) The ratio of heat transfer during processes $1 \to 2$ and $2 \to 3$ is $\left|\frac{Q_{1\to2}}{Q_{2\to3}}\right| = \frac{5}{3}$

(4) The ratio of heat transfer during processes $1 \to 2$ and $3 \to 4$ is $\left|\frac{Q_{1\to2}}{Q_{2\to4}}\right| = \frac{1}{2}$

Ans.[2,3]

Sol. (1) Isochoric and Isobaric so option (1) is wrong

(2) $W = W_{12} + W_{23} + W_{24} + W_{41}$

$= nR(2T_0 - T_0) + 0 + nR\left(\frac{T_0}{2} - T_0\right) + 0$

$= n\frac{RT_0}{2} = \frac{RT_0}{2}$ is correct

(3) $\dfrac{Q_{12}}{Q_{23}} = \dfrac{nC_PdT_{12}}{nC_VdT_{23}} = \dfrac{\frac{5R}{2}(2T_0 - T_0)}{\frac{3R}{2}(T_0 - 2T_0)} = \dfrac{5}{3}$ is correct

(4) $\dfrac{Q_{12}}{Q_{34}} = \dfrac{nC_PdT_{12}}{nC_PdT_{34}} = \dfrac{2T_0 - T_0}{\frac{T_0}{2} - T_0} = \dfrac{2}{1}$ is wrong

Q.8 Two identical moving coil galvanometers have 10 Ω resistance and full scale deflection at 2 μA current. One of them is converted into a voltmeter of 100 mV full scale reading and the other into an Ammeter of 1 mA full scale current using appropriate resistors. These are then used to measure the voltage and current in the Ohm's law experiment with R = 1000 Ω resistor by using an ideal cell. Which of the following statement(s) is/are correct ?

(1) The resistance of the Ammeter will be 0.02 Ω (round off to 2nd decimal place)

(2) The measured value of R will be 978 Ω < R < 982 Ω

(3) If the ideal cell is replaced by a cell having internal resistance of 5 Ω then the measured value of R will be more than 1000 Ω

(4) The resistance of the Voltmeter will be 100 kΩ

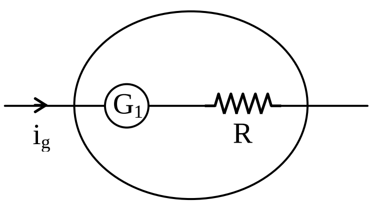
Ans.[1,2]

Sol. Resistance of galvanometer = 10 Ω

Full deflection current $i_g = 2 \times 10^{-6}$ amp.

G_1 to voltmeter

$V = 100 \times 10^{-3}$ V



Voltmeter

$V = (G_1 + R)\, i_g$

$G_1 + R = \dfrac{V}{i_g}$

$= \dfrac{10^{-1}}{2\times10^{-6}}$

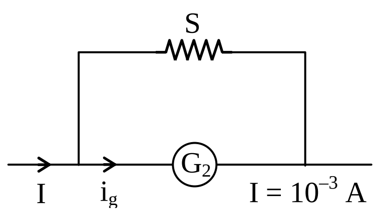
$= \dfrac{10^5}{2}$

$= 5 \times 10^4$ Ω

$\therefore R_V = 5 \times 10^4$ Ω

(4) is wrong

G_2 to Ammeter



$S = \dfrac{G.i_g}{i - i_g}$

$= \dfrac{10\times2\times10^{-6}}{10^{-3} - 2\times10^{-6}}$

$= \dfrac{10^{-6}[20]}{10^{-6}[1000-2]}$

$\approx \dfrac{20}{1000-2}$

≈ 0.02 Ω

$\therefore$ (1) is correct

(1), (2) are correct

SECTION – 3 (Maximum Marks : 18)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to TWO decimal places.
- Answer to each question will be evaluated according to the following marking scheme.

  Full Marks : +3 If ONLY the correct numerical value is entered as answer.

  Zero Marks : 0 In all other cases.

Q.1 A parallel plate capacitor of capacitance C has spacing d between two plates having area A. The region between the plates is filled with N dielectric layers, parallel to its plates, each with thickness $\delta = \frac{d}{N}$. The dielectric constant of the m^{th} layer is $K_m = K\left(1+\frac{m}{N}\right)$. For a very large N (> 10^3), the capacitance C is $\alpha\left(\frac{K\epsilon_0 A}{d\ln 2}\right)$. The value of α will be _____. [ϵ_0 is the permittivity of free space]

Sol.[1] $\left(\frac{d}{N}\right)m = x, \qquad k_m = k\left(1+\frac{x}{d}\right)$

$$dC = \frac{A\epsilon_0 k_m}{dx} = \frac{A\epsilon_0 k\left(1+\frac{x}{d}\right)}{dx}$$

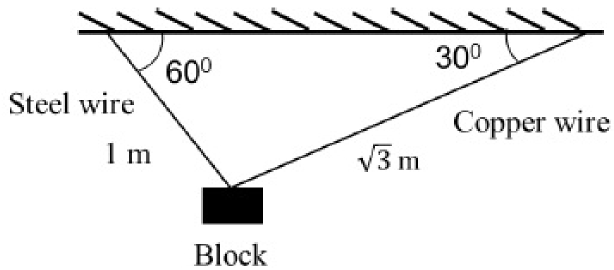
$$\frac{1}{C_{eq}} = \int_0^d \frac{dx}{A\epsilon_0 k\left(1+\frac{x}{d}\right)} = \frac{1}{A\epsilon_0 k}\int_0^d \frac{dx}{\left(1+\frac{x}{d}\right)}$$

$$\frac{1}{C_{eq}} = \frac{d}{A\epsilon_0 k}\ \ell n\left(1+\frac{x}{d}\right)_0^d \Rightarrow C_{eq} = \frac{A\epsilon_0 k}{d\ \ell n 2}$$

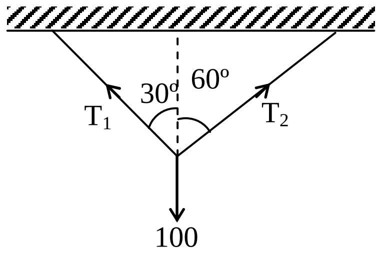
$\alpha = 1$

Q.2 A block of weight 100 N is suspended by copper and steel wires of same cross sectional area 0.5 cm^3 and, length $\sqrt{3}$ m and 1 m, respectively. Their other ends are fixed on a ceiling as shown in figure. The angles subtended by copper and steel wires with ceiling are 30º and 60º, respectively. If elongation in copper wire is (Δl_C) and elongation in steel wire is (Δl_S), then the ratio $\frac{\Delta l_C}{\Delta l_S}$ is _____.

[Young's modulus for copper and steel are 1×10^{11} N/m^2 and 2×10^{11} N/m^2, respectively]



Sol.[2]



$$\frac{T_1}{\sin 120^\circ} = \frac{T_2}{\sin 150^\circ} = \frac{100}{\sin 90^\circ}$$

$$T_1 = \frac{100\sqrt{3}}{2} = 50\sqrt{3}$$

$$T_2 = 50$$

$$y = \frac{F/A}{\Delta\ell/\ell}$$

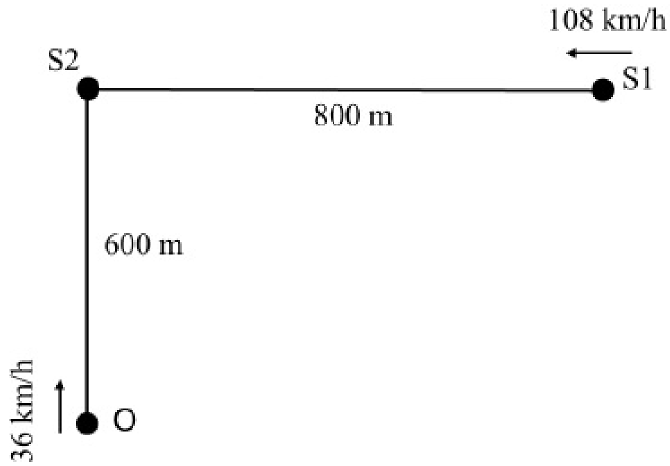
$$\Delta\ell = \frac{F\ell}{Ay}$$

$$\Delta\ell_C = \frac{T_1}{0.5 \times 2 \times 10^{11}}$$

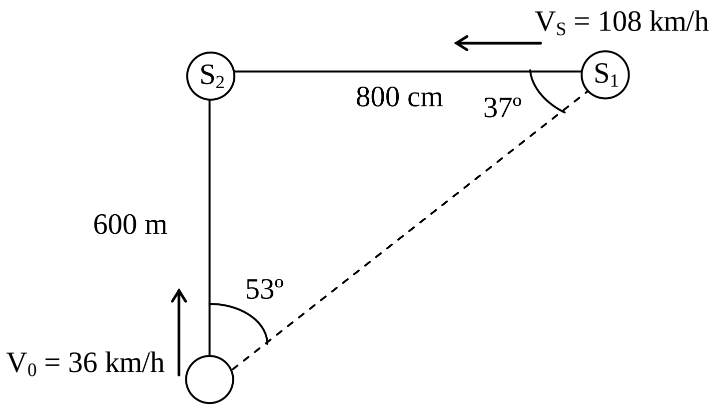
$$\frac{\Delta\ell_C}{\Delta\ell_B} = \frac{T_2\sqrt{3} \times 2}{T_1} = \frac{50\sqrt{3} \times 2}{50\sqrt{3}} = 2$$

Q.3 A train S1, moving with a uniform velocity of 108 km/h, approaches another train S2 standing on a platform. An observer O moves with a uniform velocity of 36 km/h towards S2, as shown in figure. Both the trains are blowing whistles of same frequency 120 Hz. When O is 600 m away from S2 and distance between S1 and S2 is 800 m, the number of beats heard by O is __________

[Speed of sound = 330 m/s]



Sol.[8.16]



$$n_1 = n\left(\frac{V + V_0\cos 53^o}{V - V_S\cos 37^o}\right)$$

$$= \frac{120(330 + 10 \times 3/5)}{(330 - 30 \times 4/5)}$$

$$= 120\left(\frac{336}{306}\right) = 1.098 \times 120$$

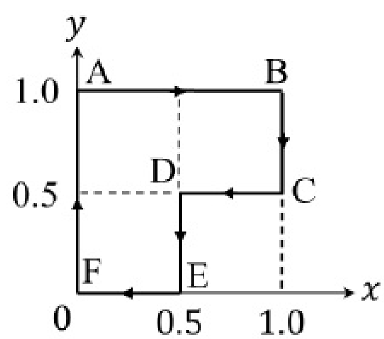
$$n_2 = n\left(\frac{V + V_0}{V}\right)$$

$$= 120\left(\frac{330 + 10}{330}\right)$$

$$= 120 \times 1.030$$

Beats = 1.0908 × 120 – 1.030 × 120 = 8.16

Q.4 A particle is moved along a path AB-BC-CD-DE-EF-FA, as shown in figure, in presence of a force $\vec{F} = (\alpha y\,\hat{i} + 2\alpha x\hat{j})\,N$, where x and y are in meter and $\alpha = -1\ Nm^{-1}$. The work done on the particle by this force $\vec{F}$ will be ______ Joule.



Sol.[0.75] A to B

$W_1 = \vec{F}.d\vec{x}$

$= \alpha y(dx)$

$= (-1)(1)(1) = -1$

B to C

$W_2 = -(2ax)\ dy$

$= -(2)(-1)(1)(0.5)$

$= +1$

$W_3 = (-1)(-0.5)(0.5)$

$= \frac{1}{4}$

$W_4 = 2(-1)(-0.5)(0.5) = \frac{1}{2}$

$W_5 = \vec{F}.d\vec{x} = 0$

$W_6 = \vec{F}.d\vec{y} = 0$

Net work = $\frac{3}{4}$ = 0.75 Joule

Q.5 A liquid at 30ºC is poured gradually in a calorimeter which is at 110ºC. Boiling temperature of liquid is 80ºC. It is found that first 5 gm of liquid is fully vapourised. After that additional 80 gm quantity of liquid is adding then equilibrium temperature is reached 50ºC. The ratio of latent and specific heats of liquid is _______. [Consider neglect heat transfer with surrounding.]

Sol.[270]

Case-I

mSdT + mL = WdT

$5 \times S \times 50 + 5L = 30\ W$

Case-II

$80 \times S \times 20 = 30\ W$

1600 S = 30 W

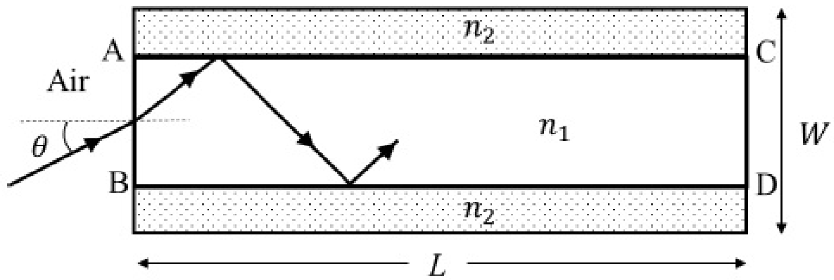
By solving

250 S + 5L = 1600 S

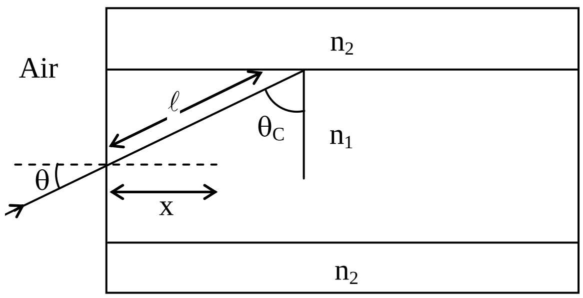
5L = 1350 S

$\frac{L}{S} = 270$

Q.6 A planar structure of length L and width W is made of two different optical media of refractive indices $n_1 = 1.5$ and $n_2 = 1.44$ as shown in figure. If L >> W, a ray entering from end AB will emerge from end CD only if the total internal reflection condition is met inside the structure. For L = 9.6 m, if the incident angle θ is varied, the maximum time taken by a ray to exit the plane CD is $t \times 10^{-9}$ s, where t is _____ [Speed of light $c = 3 \times 10^8$ m/s]



Sol.[50]



$$1.5 \sin\theta_C = 1.44 \sin 90^\circ \Rightarrow \sin\theta_C = \frac{24}{25} = \frac{x}{\ell} \Rightarrow \ell = \frac{25x}{24}$$

$$\text{Length} = \frac{25}{24} \times 9.6 \simeq 10 \text{ m}$$

$$t = \frac{10}{\frac{3\times10^8}{1.5}} = \frac{15}{3\times10^8} = 5 \times 10^{-8}$$

t = 50 ns

CAREER POINT

JEE Advanced Exam 2019
(Paper & Solution)

Date : 27 / 05 / 2019

PAPER-1

PART-I (CHEMISTRY)

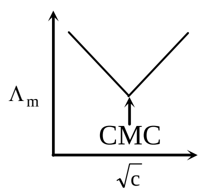
SECTION – 1 (Maximum Marks : 12)

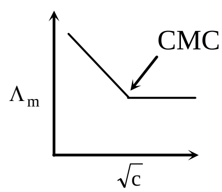
- This section contains **FOUR (04)** questions
- Each question has **FOUR** options. **ONLY ONE** of these four options is correct answer.
- For each question, choose the correct option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

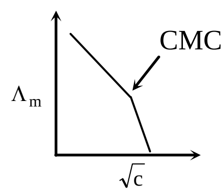
Full Marks	: **+3**	If ONLY the correct option is chosen.
Zero Marks	: **0**	If none of the options is chosen (i.e. the question is unanswered).
Negative Marks	: **–1**	In all other cases.

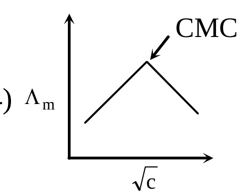
Q.1 Molar conductivity (Λ_m) of aqueous solution of sodium stearate, which behaves as a strong electrolyte, is recorded at varying concentrations (c) of sodium stearate. Which one of the following plots provides the correct representation of micelle formation in the solution ?

(critical micelle concentration (CMC) is marked with an arrow in the figures)

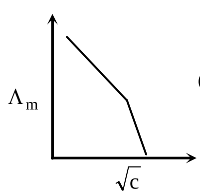
(1) 

(2) 

(3) 

(4) 

Ans. **[3]**

Sol. 

$C_{17}H_{35}COO^{-}Na^{+}$

After CMC, as the conc. increases the aggregation of sodium stearate occurs & Λ_m decreases.

Q.2 Calamine, malachite, magnetite and cryolite, respectively, are -

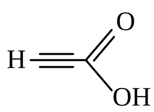
(1) $ZnCO_3$, $CuCO_3$, Fe_2O_3, Na_3AlF_6 (2) $ZnSO_4$, $CuCO_3$, Fe_2O_3, AlF_3

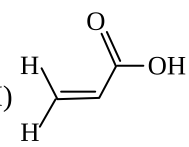
(3) $ZnSO_4$, $Cu(OH)_2$, Fe_3O_4, Na_3AlF_6 (4) $ZnCO_3$, $CuCO_3 \cdot Cu(OH)_2$, Fe_3O_4, Na_3AlF_6

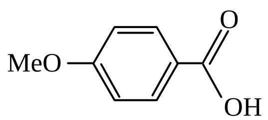
Ans. **[4]**

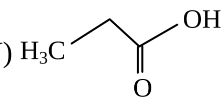
Sol. Calamine = $ZnCO_3$
Malachite = $CuCO_3 \cdot Cu(OH)_2$
Magnetite = Fe_3O_4
Cryolite = Na_3AlF_6

Q.3 The correct order of acid strength of the following carboxylic acids is –

(I) 

(II) 

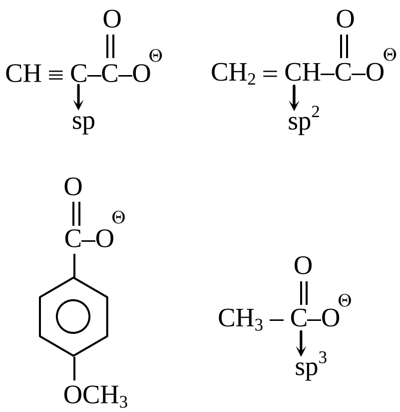
(III) 

(IV) 

(1) I > II > III > IV (2) III > II > I > IV (3) II > I > IV > III (4) I > III > II > IV

Ans. **[1]**

Sol.



I > II > III > IV

Acidic strength $\propto$ stability of conjugate anion

Q.4 The green colour produced in the borax bead test of a chromium (III) salt is due to

(1) $Cr(BO_2)_3$ (2) CrB (3) Cr_2O_3 (4) $Cr_2(B_4O_7)_3$

Ans. **[1]**

Sol. $Na_2B_4O_7 \cdot 10\,H_2O \xrightarrow{\Delta} Na_2B_4O_7 \xrightarrow{\Delta} NaBO_2 + B_2O_3$

$Cr^{3+} + B_2O_3 \rightarrow \underset{\text{Green}}{Cr\,(BO_2)_3}$

SECTION – 2 (Maximum Marks : 32)

- This section contains **EIGHT (08)** questions
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is (are) correct option(s).
- For each question, choose(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	: **+4**	If only (all) the correct option(s) is (are) chosen.
Partial Marks	: **+3**	If all the four options are correct but ONLY three options are chosen.
Partial Marks	: **+2**	If three or more options are correct but ONLY two options are chosen, both of which are correct options.
Partial Marks	: **+1**	If two or more options are correct but ONLY one option is chosen and it is a correct option.
Zero Marks	: **0**	If none of the option is chosen (i.e. the question is unanswered).
Negative Marks	: **–1**	In all other cases.

Q.1 Which of the following statement(s) is (are) true ?

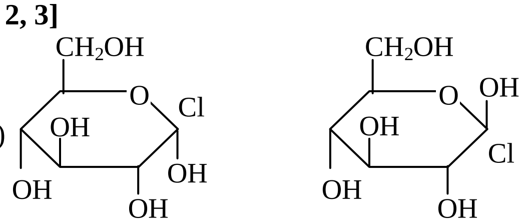
(1) The two six-membered cyclic hemiacetal froms of D–(+)– glucose are called anomers

(2) Hydrolysis of sucrose gives dextrorotatory glucose and laevorotatory fructose

(3) Monosaccharides cannot be hydrolysed to give polyhydroxy aldehydes and ketones

(4) Oxidation of glucose with bromine water gives glutamic acid

Ans. **[1, 2, 3]**

Sol. (a) 

C_1 is annomeric carbon

(b) $C_{12}H_{22}O_{11} \xrightarrow{H_2O} \underset{\text{(eavulose)}}{C_6H_{12}O_6} + \underset{\text{(Dextross)}}{C_6H_{12}O_6}$

(c) $C_6H_{12}O_6 + H_2O \rightarrow$ No reaction

(d) $C_6H_{12}O_6 + Br_2$ water $\rightarrow$ gluconic acid

Q.2 Choose the reaction(s) from the following option, for which the standard enthalpy of reaction is equal to the standard enthalpy of formation-

(1) $\frac{1}{8}S_8(s) + O_2(g) \rightarrow SO_2(g)$

(2) $2C(g) + 3H_2(g) \rightarrow C_2H_6(g)$

(3) $2H_2(g) + O_2(g) \rightarrow 2H_2O(\ell)$

(4) $\frac{3}{2}O_2(g) \rightarrow O_3(g)$

Ans. **[1,4]**

Sol. Standard Heat of Formation (1 mole)

$\frac{1}{8}S_8(s) + O_2(g) \rightarrow SO_2(g)$ it represents standard enthalpy of formation

$2C(g) + 3H_2(g) \rightarrow \underset{(g)}{C_2H_6}$ carbon is not present in elementary state

(C graphite is not present)

$\underset{(g)}{2H_2} + \underset{(g)}{O_2} \rightarrow \underset{(e)}{2H_2O}$ 2 mol product is formed

$\frac{3}{2}\underset{(g)}{O_2} \rightarrow \underset{(g)}{O_3}$ It represent standard enthalpy of formation

Q.3 An tin chloride Q undergoes the following reactions (not balanced)

$Q + Cl^- \rightarrow X$

$Q + Me_3N \rightarrow Y$

$Q + CuCl_2 \rightarrow Z + CuCl$

X is a monanion having pyramidal geometry. Both Y and Z are neutral compounds. Choose the correct options (s)

(1) The oxidation state of the central atom in Z is +2

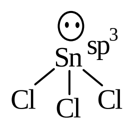
(2) There is a coordinate bond in Y

(3) The central atom in Z has one lone pair of electrons

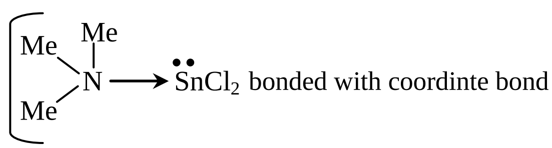
(4) The central atom in X is sp3 hybridized

Ans. **[2, 4]**

Sol. $\underset{(Q)}{SnCl_2} + Cl^- \longrightarrow \underset{(X)}{SnCl_3^-}$



$\underset{(Q)}{SnCl_2} + Me_3N \longrightarrow \underset{(X)}{SnCl_2 \cdot [NMe_3]}$



$\underset{(Q)}{SnCl_2} + CuCl_2 \longrightarrow \underset{(Z)}{SnCl_4} + CuCl$

Q.4 Fusion of MnO_2 with KOH in presence of O_2 produces a salt W. Alkaline solution of W upon electrolytic oxidation yields another salt X. The manganese containing ions present in W and X, respectively are Y and Z. Correct statement (s) is (are) -

(1) Both Y and Z are coloured and have tetrahedral shape

(2) Y is diamagnetic in nature while Z is paramagnetic

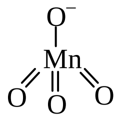
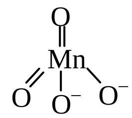
(3) In both Y and Z, π-bonding occurs between p-orbitals of oxygen and d-orbitals of manganese

(4) In aqueous acidic solution, Y undergoes disproportionation reaction to give Z and MnO_2

Ans. **[1,3,4]**

Sol. $MnO_2 \xrightarrow[\text{KOH with } O_2]{\text{Fused with}} \underset{\text{Green}}{K_2MnO_4}$ $\quad \underset{(W)}{K_2MnO_4(aq)} \rightleftharpoons 2K^+_{(aq)} + \underset{(y)}{MnO_4^{-2}(aq)}$

$K_2MnO_4 \xrightarrow[\text{oxidation in alkaline solution}]{\text{Electrolytic}} \underset{\text{Purple}}{KMnO_4}\ (X)$ Anion of X is MnO_4^- (Z)

Structure : MnO_4^-  MnO_4^{2-} 

$\underset{+7}{MnO_4^-}$ (Z) = + 7 oxidation state = 0 unpaired e^- diamagnetic

MnO_4^{2-} (Y) = + 6 oxidation state = 1 unpaired e^- paramagnetic

In acidic solution $3MnO_4^{-2}(aq) + 4H^+ \rightarrow \underset{(Z)}{2MnO_4^-} + MnO_2 + 2H_2O$

Q.5 Which of the following statement(s) is (are) correct regarding the root mean square speed (U_{rms}) and average translational kinetic energy (ε_{av}) of a molecule in a gas at equilibrium ?

(1) ε_{av} is doubled when its temperature is increased four times

(2) ε_{av} at a given temperature does not depend on its molecular mass

(3) U_{rms} is doubled when its temperature is increased four times

(4) U_{rms} is inversely proportional to the square root of its molecular mass

Ans. **[2, 3, 4]**

Sol. $E_{av} = \frac{3}{2}KT$

$U_{rms} = \sqrt{\frac{3RT}{m}}$

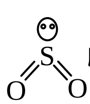
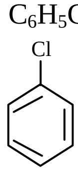
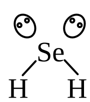
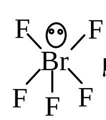
Q.6 Each of the following option contains a set of four molecules, Identify the option(s) where all four molecules posses permanent dipole moment at room temperature -

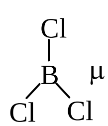
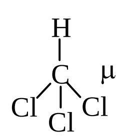
(1) SO_2, C_6H_5Cl, H_2Se, BrF_5 (2) $BeCl_2$, CO_2, BCl_3, $CHCl_3$

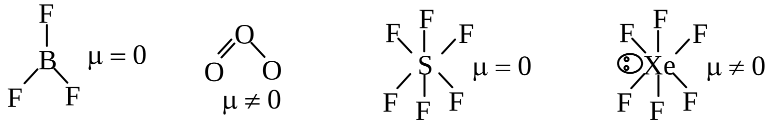
(3) BF_3, O_3, SF_6, XeF_6 (4) NO_2, NH_3, $POCl_3$, CH_3Cl

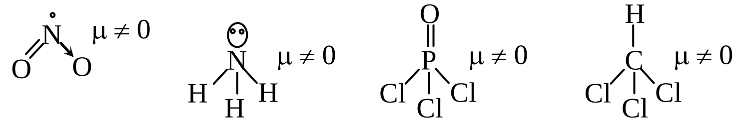
Ans. **[1, 4]**

Sol. (1) SO_2, C_6H_5Cl, H_2Se, BrF_5

 $\mu \neq 0$  $\mu \neq 0$  $\mu \neq 0$  $\mu \neq 0$

(2) $\underset{\mu=0}{Cl-Be-Cl}$, $\underset{\mu=0}{O=C=0}$,  $\mu = 0$  $\mu \neq 0$

(3) 

(4) 

Q.7 In the decay sequence

$$^{238}_{92}U \xrightarrow{-X_1} {}^{234}_{90}Th \xrightarrow{-X_2} {}^{234}_{91}Pa \xrightarrow{-X_3} {}^{234}Z \xrightarrow{-X_4} {}^{230}_{90}Th$$

x_1, x_2, x_3 and x_4 are particles / radiation emitted by the respective isotopes. The correct option(s) is(are)

(1) x_3 is γ-ray

(2) x_2 is β^-

(3) Z is an isotope of uranium

(4) x_1 will deflect towards negatively charged plate

Ans. **[2, 3, 4]**

Sol.

$_{92}U^{238} \rightarrow {}_{90}Th^{234} + {}_2He^4$ (x_1)

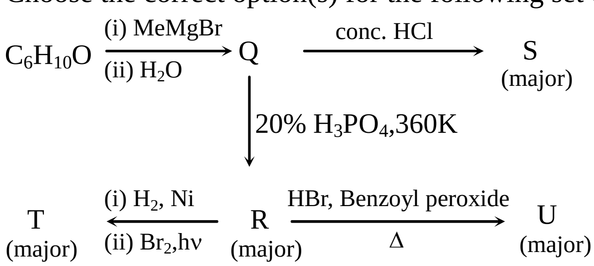
$_{90}Th^{234} \longrightarrow {}_{91}Pa^{234} + \beta^-$ or $_{-1}e^0$ (x_2)

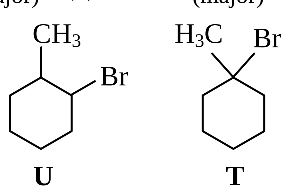
$_{91}Pa^{234} \longrightarrow {}_{92}Z^{234} + \beta^-$ or $_{-1}e^0$ (x_3)

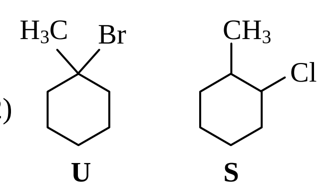
$_{92}Z^{234} \longrightarrow {}_{90}Th^{230} + {}_2He^4$ (x_4)

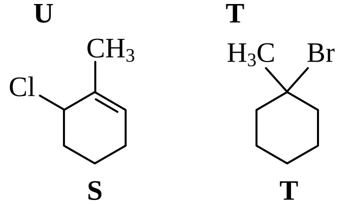
$x_1 = \alpha$
$x_2 = \beta$
$x_3 = \beta$
$x_4 = \alpha$

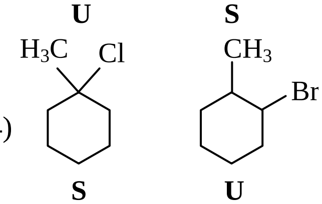
Q.8 Choose the correct option(s) for the following set of reactions



(1) 

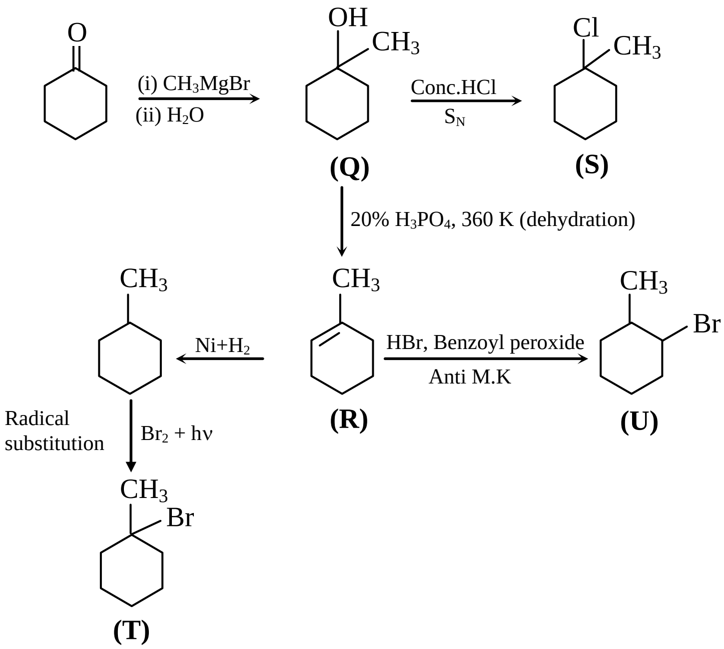
(2) 

(3) 

(4) 

Ans. [1,4]

Sol.



SECTION – 3 (Maximum Marks : 18)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value ot TWO decimal places.
- Answer to each question will be evaluated according to the following marking scheme.

  Full Marks : +3 If ONLY the correct numerical value is entered as answer.

  Zero Marks : 0 In all other cases.

Q.1 Consider the kinetic data given in the following table for the reaction A + B + C → Product

Experiment No.	[A] (mol dm^{-3})	[B] (mol dm^{-3})	[C] (mol dm^{-3})	Rate of reaction (mol dm^{-3} s^{-1})
1	0.2	0.1	0.1	6.0×10^{-5}
2	0.2	0.2	0.1	6.0×10^{-5}
3	0.2	0.1	0.2	1.2×10^{-4}
4	0.3	0.1	0.1	9.0×10^{-5}

The rate of the reaction for [A] = 0.15 mol dm^{-3}, [B] = 0.25 mol dm^{-3} and [C] = 0.15 mol dm^{-3} is found to be Y × 10^{-5} mol dm^{-3} s^{-1}. The value of Y is _____ .

Ans. [6.75]

Sol. $r = K[A]^b[B]^q[C]^n$

$$\frac{r_2}{r_1} = \left[\frac{B_2}{B_1}\right]^q$$

$1 = 2^q$ or $2^0 = 2^q$ $\therefore q = 0$

$$\frac{r_3}{r_2} = \left[\frac{C_3}{C_2}\right]^n$$

$2 = 2^n$ $n = 1$

$$\frac{r_4}{r_1} = \left[\frac{A_4}{A_1}\right]^p \left[\frac{C_4}{C_1}\right]^n$$

$$\frac{9}{6} = \left(\frac{3}{2}\right)^p \times \left(\frac{3}{2}\right)^1 \Rightarrow P = 1$$

$r = K[A]^1[C]^1$

$$K = \frac{r}{[A][C]} = \frac{6\times10^{-5}}{2\times10^{-2}} = 3\times10^{-3}$$

$r = K[A][C]$

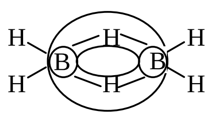
$= 3\times10^{-3} \times 0.15 \times 0.15$

$= 6.75 \times 10^{-5}$

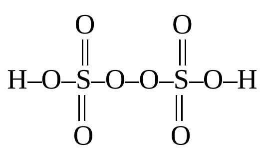
Q.2 Among B_2H_6, $B_3N_3H_6$, N_2O, N_2O_4, $H_2S_2O_3$ and $H_2S_2O_8$ the total number of molecules containing covalent bond between two atoms of the same kind is ____

Ans. [4]

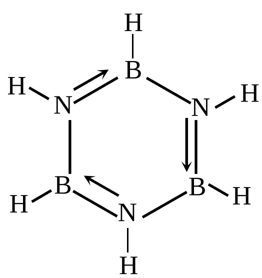
Sol. B_2H_6



$H_2S_2O_8$

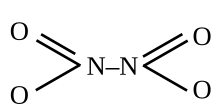


$B_3N_3H_6$

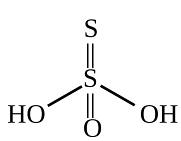


N_2O $N \equiv N \rightarrow O$

N_2O_4



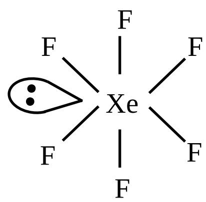
$H_2S_2O_3$



Q.3 At 143 K, the reaction of XeF_4 with O_2F_2 produces a xenon compound Y. The total number of lone pair(s) of electron present on the whole molecule of Y is ____

Ans. **[19]**

Sol. $$XeF_4 + O_2F_2 \longrightarrow \underset{(Y)}{XeF_6} + O_2$$



Total no. of lone pair present on the whole molecule of Y = 1 ℓ. p. in Xe + 18. ℓ. p. in F

= 19 total ℓ. p.

Q.4 On dissolving 0.5 g of a non-volatile non-ionic solute to 39 g of benzene, its vapor pressure decreases from 650 mm Hg to 640 mm Hg. The depression of freezing point of benzene (in K) upon addition of the solute is _____

(Given data : Molar mass and the molal freezing point depression constant of benzene are 78 g mol^{-1} and 5.12 K kg mol^{-1}, respectively)

Ans. [1.03]

Sol. $$\frac{P_B^0 - P_S}{P_S} = \frac{n_A}{n_B}$$

$$\frac{650 - 640}{640} = \frac{n_A}{0.5}$$

$$\frac{10 \times 0.5}{640} = n_A = \frac{5}{640}$$

$$\Delta T_f = iK_f\, m$$

$$= (1)\ (5.12)\ \frac{5 \times 1000}{640 \times 39}$$

= 1.0256 = 1.026

= 1.03

Q.5 For the following reaction the equilibrium constant K_c at 298 K is 1.6×10^{17}

$$Fe^{2+}(aq) + S^{2-}(aq) \rightleftharpoons FeS(s)$$

When equal volumes of 0.06 M Fe^{2+} (aq) and 0.2 M S^{2-} (aq) solutions are mixed, the equilibrium concentration of Fe^{2+} (aq) is found to be **Y** $\times 10^{-17}$ M. The value of Y is _____

Ans. [8.93]

Sol.

	Fe^{2+} +	S^{2-}	$\rightleftharpoons$ FeS(s)
before mixing	0.06M	0.02 M	
after mixing	0.03 M	0.1 M	

after reaction δ 0.07 M as K_c is very high reaction proceed towards completion

(very small) There fore limiting reagent will be consumed almost completely

Since,

$K_c = K_c = 1.6 \times 10^{17}$ (Very high value)

$$Kc = \frac{1}{[Fe^{+2}][S^{-2}]}$$

$$\delta = [Fe^{+2}] = \frac{1}{[K_c][S^{-2}]}$$

$$= \frac{1}{(1.6\times10^{17})\ (0.07)}$$

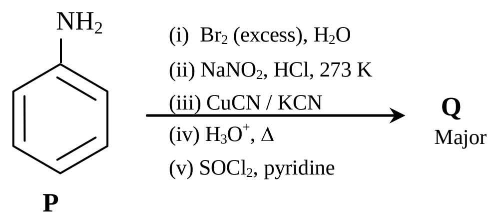
$$= \frac{100}{7\times1.6}\times10^{-17}$$

$= 8.93 \times10^{-17}$

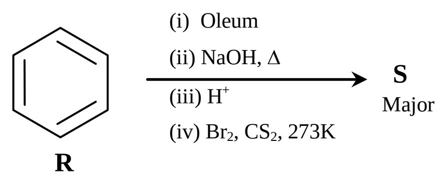
Y = 8.93

Q.6 Schemes 1 and 2 describe the conversion of P to Q and R to S, respectively. Scheme 3 describes the synthesis of T from Q and S. The total number of Br atoms in a molecule of T is _____

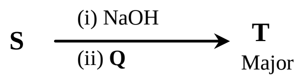
Scheme 1 :



Scheme 2 :

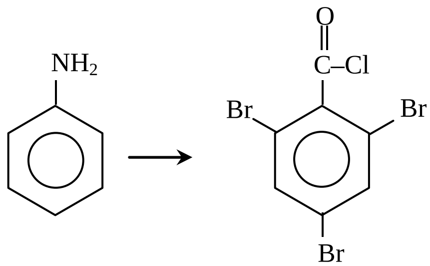


Scheme 3 :



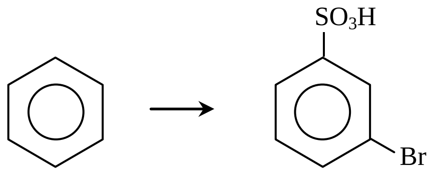
Ans. [4]

Sol. **Scheme 1 :**



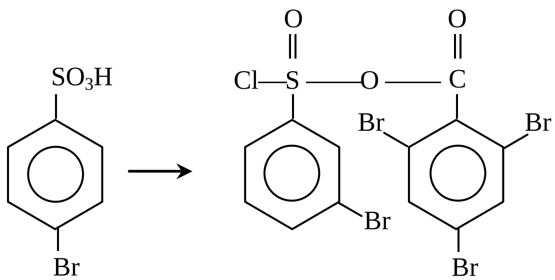
(Q)

Scheme 2 :



(S)

Scheme 3 :



CAREER POINT

JEE Advanced Exam 2019

(Paper & Solution)

Date : 27 / 05 / 2019

PAPER-1

PART-I (MATHEMATICS)

SECTION – 1 (Maximum Marks : 12)

- This section contains **FOUR (04)** questions
- Each question has **FOUR** options. **ONLY ONE** of these four options is the correct answer.
- For each question, choose the option corresponding to the correct answer.
- Answer to each question will be evaluated according to the following marking scheme :

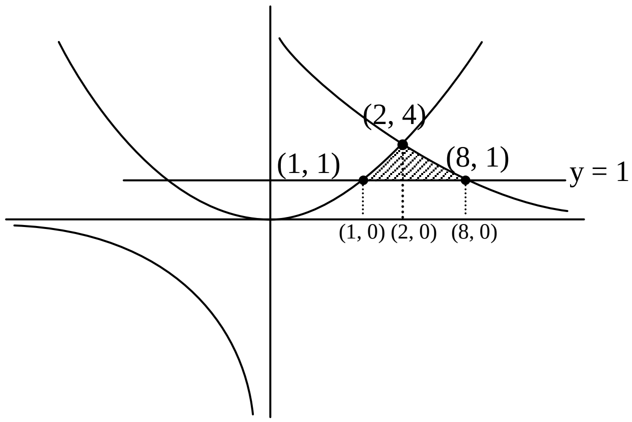
Full Marks	: **+3**	If ONLY the correct option is chosen.
Zero Marks	: **0**	If none of the options is chosen (i.e. the question is unanswered).
Negative Marks	: **–1**	In all other cases.

Q.1 The area of the region $\{(x, y) : xy \le 8, 1 \le y \le x^2\}$ is -

(1) $8 \log_e 2 - \frac{14}{3}$ (2) $16 \log_e 2 - \frac{14}{3}$ (3) $16 \log_e 2 - 6$ (4) $8 \log_e 2 - \frac{7}{3}$

Ans. **[2]**

Sol.



$$\text{Required area} = \int_1^2 x^2 dx + \int_2^8 \left(\frac{8}{x}\right) dx - (7)$$

$$= \left[\frac{x^3}{3}\right]_1^2 + 8(\ell n\, x)_2^8 - 7$$

$= \frac{8}{3} - \frac{1}{3} + 8\,(\ell n\, 8 - \ell n\, 2) - 7$

$= \frac{7}{3} + 8\,(2\,\ell n\, 2) - 7$

$= -\frac{14}{3} + 16\,\ell n\, 2$

Q.2 Let $M = \begin{bmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{bmatrix} = \alpha I + \beta M^{-1}$, where $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ are real numbers, and I is the 2×2 identity matrix. If α^* is the minimum of the set $\{\alpha(\theta) : \theta \in [0, 2\pi)\}$ and β^* is the minimum of the set $\{\beta(\theta) : \theta \in [0, 2\pi)\}$, then value of $\alpha^* + \beta^*$ is -

(1) $-\frac{37}{16}$ (2) $-\frac{31}{16}$ (3) $-\frac{17}{16}$ (4) $-\frac{29}{16}$

Ans. **[4]**

Sol. $m = \sin^4\theta\cos^4\theta + (1 + \sin^2\theta)(1 + \cos^2\theta)$

$m = 2 + \sin^4\theta\cos^4\theta + \sin 2\theta\cos^2\theta$

$$\begin{bmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} + \frac{\beta}{|M|}\begin{bmatrix} \cos^4\theta & 1+\sin^2\theta \\ -1-\cos^2\theta & \sin^4\theta \end{bmatrix}$$

$\sin^4\theta = \alpha + \frac{\beta}{|M|}\cos^4\theta$ …(1)

$-1 - \sin^2\theta = \frac{\beta}{|M|}(1 + \sin^2\theta)$ … (2)

$1 + \cos^2\theta = \frac{\beta}{|M|}(-1 - \cos^2\theta)$ … (3)

$\cos^4\theta = \alpha + \frac{\beta}{|M|}\sin^4\theta$ …(4)

From eq^n (3) $\beta = -|M|$

$\beta = -(\sin^4\theta\cos^4\theta + \sin^2\theta\cos^2\theta + 2)$

$\beta = -(t^2 + t + 2)$ $t = \sin^2\theta\cos^2\theta$

$t = \frac{1}{4}(\sin 2\theta)^2$

$0 \le t \le \frac{1}{4}$

β_{min} at $t = \frac{1}{4}$ $\beta_{min} = -\left(\frac{1}{16} + \frac{1}{4} + 2\right)$

$\beta_{min} = -\frac{1+4+32}{16}$

$\beta_{min} = -\frac{37}{16}$

From (1)

$\sin^4\theta = \alpha - \cos^4\theta$

$\alpha = \sin^4\theta + \cos^4\theta$

$\alpha = 1 - \frac{1}{2}(\sin^2 2\theta)$

$\alpha_{min} = \frac{1}{2}$

$\alpha_{min} + \beta_{min} = \frac{-37}{16} + \frac{1}{2} = \frac{-29}{16}$ Option (4) is correct

Q.3 A line y = mx + 1 intersects the circle $(x - 3)^2 + (y + 2)^2 = 25$ at the points P and Q. If the midpoint of the line segment PQ has x-coordinate $-\frac{3}{5}$, then which one of the following options is correct ?

(1) $-3 \le m < -1$ (2) $2 \le m < 4$ (3) $4 \le m < 6$ (4) $6 \le m < 8$

Ans. **[2]**

Sol. Coordinate of mid point of chord

$\left(\frac{-3}{5}, -\frac{3m}{5} + 1\right)$

Slope of line joining mid point of chord with centre (3, –2)

$= \frac{-\frac{3m}{5} + 1 + 2}{-\frac{3}{5} - 3} = \frac{-3m + 15}{-18}$

Now this line is perpendicular to given chord

$\left(\frac{-3m + 15}{-18}\right)(m) = -1$

$-3m^2 + 15m = 18$

$3m^2 - 15m + 18 = 0$

$m^2 - 5m + 6 = 0$

$(m - 2)(m - 3) = 0$

m = 2, m = 3

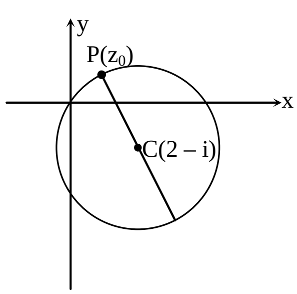
option (2) is correct $2 \le m \le 4$

Q.4 Let S be the set of all complex numbers z satisfying $|z - 2 + i| \ge \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0 - 1|}$ is the maximum of the set $\left\{\frac{1}{|z - 1|} : z \in S\right\}$, then the principal argument of $\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2i}$ is -

(1) $\frac{\pi}{2}$ (2) $\frac{\pi}{4}$ (3) $-\frac{\pi}{2}$ (4) $\frac{3\pi}{4}$

Ans. **[3]**

Sol.



$|z-(2-i)| \geq \sqrt{5}$

for $|z_0 - 1|$ to be minimum

$z_0 = x_0 + iy_0$ is at point P as shown in figure

$\arg\left(\frac{4-(z_0+\bar{z}_0)}{z_0-\bar{z}_0+2i}\right) = \arg\left(\frac{4-2x}{2iy+2i}\right)$

$= \arg\left(\frac{-i(2-x)}{y+2}\right)$

$= \arg(-i\lambda) = -\frac{\pi}{2}$ option (3) is correct

SECTION – 2 (Maximum Marks : 32)

- This section contains **EIGHT (08)** questions
- Each question has **FOUR** options. **ONE OR MORE THAN ONE** of these four option(s) is (are) correct answer(s).
- For each question, choose the option(s) corresponding to (all) the correct answer(s).
- Answer to each question will be evaluated according to the following marking scheme :

Full Marks	: **+4**	If only (all) the correct option(s) is (are) chosen.
Partial Marks	: **+3**	If all the four options are correct but ONLY three options are chosen.
Partial Marks	: **+2**	If three or more options are correct but ONLY two options are chosen, both of which are correct.
Partial Marks	: **+1**	If two or more options are correct but ONLY one option is chosen and it is a correct option.
Zero Marks	: **0**	If none of the option is chosen (i.e. the question is unanswered).
Negative Marks	: **–1**	In all other cases.

- For example, in a question, if (A), (B) and (D) are the ONLY three options corresponding to correct answers, then
  choosing ONLY (A), (B) and (D) will get +4 marks;
  choosing ONLY (A) and (B) will get +2 marks;
  choosing ONLY (A) and (D) will get +2 marks;
  choosing ONLY (B) and (D) will get +2 marks;
  choosing ONLY (A) will get +1 marks;
  choosing ONLY (B) will get +1 marks;
  choosing ONLY (D) will get +1 marks;
  choosing no option (i.e. the question is unanswered) will get 0 marks; and
  choosing any other combination of options will get –1 mark.

Q.1 Let α and β be the roots of $x^2 - x - 1 = 0$, with $\alpha > \beta$. For all positive integers n, define

$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$, $n \geq 1$

$b_1 = 1$ and $b_n = a_{n-1} + a_{n+1}$, $n \geq 2$.

Then which of the following options is/are correct ?

(1) $b_n = \alpha^n + \beta^n$ for all $n \geq 1$

(2) $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$

(3) $a_1 + a_2 + a_3 + \ldots\ldots + a_n = a_{n+2} - 1$ for all $n \geq 1$

(4) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

Ans. **[1,2,3]**

Sol. $x^2 - x - 1 = 0$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad b = \frac{1-\sqrt{5}}{2}$$

(1) $b_n = a_{n-1} + a_{n-1}$

$$b_n = \frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} + \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$$

$$b_n = \frac{\alpha^{n-1} - \beta^{n-1} + \alpha^{n+1} - \beta^{n+1}}{(\alpha - \beta)}$$

$$b_n = \frac{\alpha^{n-1}(\alpha^2 + 1) - \beta^{n-1}(\beta^2 + 1)}{(\alpha - \beta)}$$

$$b_n = \frac{\alpha^{n-1}(\alpha + 2) - \beta^{n-1}(\beta + 2)}{(\alpha - \beta)}$$

$$b_n = \frac{\alpha^{n-1}\left(\frac{5+\sqrt{5}}{2}\right) - \beta^{n-1}\left(\frac{5-\sqrt{5}}{2}\right)}{(\alpha - \beta)}$$

$$b_n = \frac{\sqrt{5}\,\alpha^{n-1}\left(\frac{\sqrt{5}+1}{2}\right) - \sqrt{5}\,\beta^{n-1}\left(\frac{\sqrt{5}-1}{2}\right)}{(\alpha - \beta)}$$

$$b_n = \frac{\sqrt{5}\,\alpha^{n-1}(\alpha) - \sqrt{5}\,\beta^{n-1}(-\beta)}{\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}}$$

$$b_n = \frac{\sqrt{5}\alpha^n + \sqrt{5}\beta^n}{\sqrt{5}}$$

$b_n = \alpha^n + \beta^n$ Option (1) is correct

(2) $$\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \sum_{n=1}^{\infty} \frac{\alpha^n - \beta^n}{(\alpha - \beta)10^n}$$

$$= \frac{1}{\sqrt{5}} \sum_{n=1}^{\infty} \left(\frac{\alpha}{10}\right)^n - \left(\frac{\beta}{10}\right)^n$$

$$= \frac{1}{\sqrt{5}} \left[\frac{\frac{\alpha}{10}}{1 - \frac{\alpha}{10}} - \frac{\frac{\beta}{10}}{1 - \frac{\beta}{10}} \right]$$

$$= \frac{1}{\sqrt{5}} \left(\frac{\alpha}{10 - \alpha} - \frac{\beta}{10 - \beta} \right)$$

$$= \frac{1}{\sqrt{5}} \left(\frac{\alpha(10 - \beta) - \beta(10 - \alpha)}{(10 - \alpha)(10 - \beta)} \right)$$

$$= \frac{1}{\sqrt{5}} \left(\frac{10\alpha - \alpha\beta - 10\beta + \alpha\beta}{100 - 10(\alpha + \beta) + \alpha\beta} \right)$$

$$= \frac{1}{\sqrt{5}}\frac{10(\alpha-\beta)}{100-10(\alpha+\beta)+\alpha\beta}$$

$$= \frac{10}{100-10-1} = \frac{10}{89}$$ option (2) is correct

(3) $a_1 + a_2 + a_3 + ... + a_n$

$$= \sum_{i=1}^{n} a_i = \sum_{i=1}^{n}\frac{\alpha^i-\beta^i}{\alpha-\beta}$$

$$= \frac{1}{(\alpha-\beta)}\left(\frac{\alpha(1-\alpha^n)}{(1-\alpha)} - \frac{\beta(1-\beta^n)}{(1-\beta)}\right)$$

$$= \frac{(\alpha+1)(1-\alpha^n)-(\beta+1)(1-\beta^n)}{(1-\alpha)(1-\beta)(\alpha-\beta)}$$

$$= \frac{\alpha^2-\alpha^{n+2}-\beta^2+\beta^{n+2}}{(1-\alpha)(1-\beta)(\alpha-\beta)}$$

$$= \frac{\sqrt{5}+\beta^{n+2}-\alpha^{n+2}}{(\beta-\alpha)} = -1 + a_{n+2}$$ option (3) is correct

(4) $$\sum_{n=1}^{\infty}\frac{b_n}{10^n} = \sum_{n=1}^{\infty}\left(\frac{\alpha}{10}\right)^n + \sum_{n=1}^{\infty}\left(\frac{\beta}{10}\right)^n$$

$$= \frac{\frac{\alpha}{10}}{1-\frac{\alpha}{10}} + \frac{\frac{\beta}{10}}{1-\frac{\beta}{10}} = \frac{\alpha}{10-\alpha} + \frac{\beta}{10-\beta}$$

$$= \frac{10(\alpha+\beta)-2\alpha\beta}{100-10(\alpha+\beta)+\alpha\beta} = \frac{10+2}{89} = \frac{12}{89}$$ option (4) is wrong

Q.2 Let L_1 and L_2 denote the lines $\vec{r} = \hat{i}+\lambda(-\hat{i}+2\hat{j}+2\hat{k}),\ \lambda \in R$ and $\vec{r} = \mu(2\hat{i}-\hat{j}+2\hat{k})$, $\mu \in R$ respectively. If L_3 is a line which is perpendicular to both L_1 and L_2 and cuts both of them, then which of the following options describe(s) L_3?

(1) $\vec{r} = \frac{1}{3}(2\hat{i}+\hat{k})+t(2\hat{i}+2\hat{j}-\hat{k}),\ t \in R$

(2) $\vec{r} = t(2\hat{i}+2\hat{j}-\hat{k}), t \in R$

(3) $\vec{r} = \frac{2}{9}(4\hat{i}+\hat{j}+\hat{k}) + t(2\hat{i}+2\hat{j}-\hat{k}),\ t \in R$

(4) $\vec{r} = \frac{2}{9}(2\hat{i}-\hat{j}+2\hat{k}) + t(2\hat{i}+2\hat{j}-\hat{k}), t \in R$

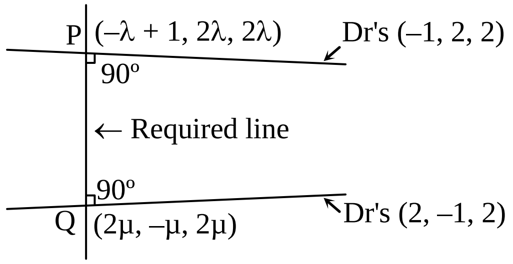
Ans. **[1,3,4]**

Sol. Line (1) : $\vec{r} = \hat{i}+\lambda(-\hat{i}+2\hat{j}+2\hat{k}),\ \lambda \in R$

Line (2) : $\vec{r} = \mu(2\hat{i}-\hat{j}+2\hat{k})$, $\mu \in R$

Line (1) $\frac{x-1}{-1} = \frac{y-0}{2} = \frac{z-0}{2} = \lambda$

Line (2) $\frac{x-0}{2} = \frac{y-0}{-1} = \frac{z-0}{2} = \mu$



Dr's of line PQ $(-\lambda - 2\mu + 1, 2\lambda + \mu, 2\lambda - 2\mu)$

Let first line parallel to $\vec{a} = -\hat{i} + 2\hat{j} + 2\hat{k}$

Let second line parallel to $\vec{b} = 2\hat{i} - \hat{j} + 2\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= \hat{i}(6) - \hat{j}(-2-4) + \hat{k}(1-4)$$

$$= 6\hat{i} + 6\hat{j} - 3\hat{k}$$

So that Dr's of required line (2, 2, –1)

Now $\dfrac{-\lambda - 2\mu + 1}{2} = \dfrac{2\lambda + \mu}{2} = \dfrac{2\lambda - 2\mu}{-1}$

$-\lambda - 2\mu + 1 = 2\lambda + \mu$

$3\lambda + 3\mu = 1$ …(1)

$-2\lambda - \mu = 4\lambda - 4\mu$

$6\lambda = 3\mu$

$\mu = 2\lambda$ …(2)

From (1) & (2)

$\lambda = \dfrac{1}{9}$

$\mu = \dfrac{2}{9}$

Point P $\left(\dfrac{8}{9}, \dfrac{2}{9}, \dfrac{2}{9}\right)$

Point Q $\left(\dfrac{4}{9}, \dfrac{-2}{9}, \dfrac{4}{9}\right)$

Now equation of required line

$\vec{r} = \left(\dfrac{8}{9}\hat{i} + \dfrac{2}{9}\hat{j} + \dfrac{2}{9}\hat{k}\right) + t(2\hat{i} + 2\hat{j} - \hat{k}) \rightarrow$ option (3) correct

$\vec{r} = \left(\dfrac{4}{9}\hat{i} - \dfrac{2}{9}\hat{j} + \dfrac{4}{9}\hat{k}\right) + t(2\hat{i} + 2\hat{j} - \hat{k}) \rightarrow$ option (4) correct

option 2 is wrong

option (1) is correct

Q.3 Let $f : R \to R$ be given by $f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1, & x < 0; \\ x^2 - x + 1, & 0 \le x < 1; \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \le x < 3; \\ (x-2)\log_e(x-2) - x + \frac{10}{3}, & x \ge 3 \end{cases}$

Then which of the following options is/are correct ?

(1) f is increasing on $(-\infty, 0)$

(2) f is onto

(3) f' has a local maximum at x = 1

(4) f' is NOT differentiable at x = 1

Ans. **[2,3,4]**

Sol.

$$f(x) = \begin{Bmatrix} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1 & x < 0 \\ x^2 - x + 1 & 0 \le x < 1 \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3} & 1 \le x < 3 \\ (x-2)\log_e(x-2) - x + \frac{10}{3} & x \ge 3 \end{Bmatrix}$$

$$f'(x) = \begin{Bmatrix} 5(x+1)^4 - 2 & x < 0 \\ 2x - 1 & 0 \le x < 1 \\ 2x^2 - 8x + 7 & 1 \le x < 3 \\ \ell n(x-2) & x \ge 3 \end{Bmatrix}$$

$x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1$ takes value between $-\infty$ to 1

also $(x-2)\ \ell n(x-2) - x + \frac{10}{3}$ takes value between $\frac{1}{3}$ to ∞

So range of f(x) is R.

option (1) is correct

$f''(1^-) = 2$ and $f''(1^+) = -4$

so $f'(x)$ is not diff. at x = 1

so option (2) is correct

$f'(x)$ has local maxima at x = 1

so option (3) is correct

Q.4 Let $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$ and adj $M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$ where a and b are real numbers. Which of the following options is/are correct ?

(1) If $M \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $\alpha - \beta + \gamma = 3$

(2) a + b = 3

(3) $(\text{adj } M)^{-1} + \text{adj } M^{-1} = -M$

(4) $\det(\text{adj } M^2) = 81$

Ans. **[1,2,3]**

Sol. $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$ $\quad$ $\text{adj } M = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$

$2 - 3b = -1$ $\quad$ $3 - 2a = -1$

$3b = 3$ $\quad$ $-2a = -4$

$b = 1$ $\quad$ $a = 2$

Option (1)

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\beta + 2\gamma = 1$

$\alpha + 2\beta + 3\gamma = 2$

$3\alpha + \beta + \gamma = 3$

Solve it

$\alpha = 1, \beta = -1, \gamma = 1$

$\alpha - \beta + \gamma = 1 + 1 + 1 = 3$ $\quad$ option (1) correct

Option (2)

$a + b = 1 + 2 = 3$

Option (3) $(\text{adj } M)^{-1}$ adj M^{-1}

$= 2 \text{ adj } (M^{-1})$

$= 2\left(-\dfrac{M}{2}\right)$

$= -M$

Option (4)

$|\text{adj } M^2| = |M|^2 = |M|^4 = 16$

Q.5 There are three bags B_1, B_2 and B_3. The bag B_1 contains 5 red and 5 green balls, B_2 contains 3 red and 5 green balls, and B_3 contains 5 red and 3 green balls. Bags B_1, B_2 and B_3 have probabilities $\frac{3}{10}$, $\frac{3}{10}$ and $\frac{4}{10}$ respectively of being chosen. A bag is selected at random and a ball is chosen at random from the bag. Then which of the following options is/are correct ?

(1) Probability that the selected bag is B_3, given that the chosen ball is green, equals $\frac{5}{13}$

(2) Probability that the chosen ball is green, given that the selected bag is B_3, equals $\frac{3}{8}$

(3) Probability that the selected bag is B_3 and the chosen ball is green equals $\frac{3}{10}$

(4) Probability that the chosen ball is green equals $\frac{39}{80}$

Ans. **[2 & 4]**

Sol.

Bag I: 5G, 5R

Bag II: 5G, 3R

Bag III: 3G, 5R

Option (1) $P\left(\frac{B_3}{G}\right) = \frac{P(B_3 \cap G)}{P(G)} = \frac{\frac{4}{10}\times\frac{3}{8}}{39/80} = \frac{12}{39} = \frac{4}{13}$

Option (2) $P\left(\frac{G}{B_3}\right) = \frac{P(G \cap B_3)}{P(B_3)} = \frac{P(G)P(B_3)}{P(B_3)} = P(G) = \frac{3}{8}$

Option (3) $P(B_3 \cap G) = P(B_3)\,P(G) = \frac{4}{10} \times \frac{3}{8} = \frac{3}{20}$

Option (4) $P(G) = P(B_1)\,P\left(\frac{G}{B_1}\right) + P(B_2)\,P\left(\frac{G}{B_2}\right) + P(B_3)\,P\left(\frac{G}{B_3}\right)$

$= \frac{3}{10}\times\frac{5}{10} + \frac{3}{10}\times\frac{5}{8} + \frac{4}{10}\times\frac{3}{8}$

$= \frac{3}{20} + \frac{15}{80} + \frac{12}{80}$

$= \frac{12+15+12}{80} = \frac{39}{80}$

Option (2) & (4) correct

Q.6 Let Γ denote a curve y = y(x) which is in the first quadrant and let the point (1, 0) lie on it. Let the tangent to Γ at a point P intersect the y-axis at Y_P. If PY_P has length 1 for each point P on Γ, then which of the following options is/are correct ?

(1) $xy' + \sqrt{1-x^2} = 0$

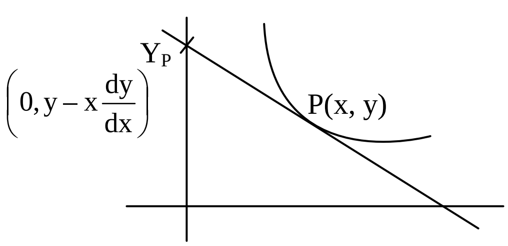
(2) $y = \log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$

(3) $xy' - \sqrt{1-x^2} = 0$

(4) $y = -\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2}$

Ans. **[1,2]**

Sol.



equation of tangent

$$(Y - y) = \frac{dy}{dx}(X - x)$$

Put X = 0

$$Y = y - x.\frac{dy}{dx}$$

Distance between $PY_P = 1$

$$\sqrt{x^2 + x^2\left(\frac{dy}{dx}\right)^2} = 1$$

$$x^2 + x^2\left(\frac{dy}{dx}\right)^2 = 1$$

$$x^2\left(\frac{dy}{dx}\right)^2 = 1 - x^2$$

$$x\frac{dy}{dx} = \pm\sqrt{1-x^2}$$

$$x\frac{dy}{dx} = \sqrt{1-x^2}$$

$$\int dy = \int \frac{\sqrt{1-x^2}}{x}dx$$

put $x = \sin\theta$

$dx = \cos\theta\, d\theta$

$$y = \int \frac{\cos^2\theta}{\sin\theta}d\theta$$

$$y = \int \frac{1-\sin^2\theta}{\sin\theta}d\theta$$

$$x\frac{dy}{dx} = -\sqrt{1-x^2}$$

$$dy = -\int \frac{\sqrt{1-x^2}}{x}dx$$

Integrate

$$y = \ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2} + c$$

$f(1) = 0, \quad c = 0$

$$y = \ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$$

$$y = \int (\operatorname{cosec}\theta - \sin\theta)d\theta$$

$$y = \ell n(\operatorname{cosec}\theta - \cot\theta) + \cos\theta + c$$

$$y = \ell n\left(\frac{1}{x} - \frac{\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2} + c$$

$$y = \ell n\left(\frac{1-\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2} + c$$

$$y = \ell n\left(\frac{x^2}{1+\sqrt{1-x^2}} \times \frac{1}{x}\right) + \sqrt{1-x^2} + c$$

$$y = -\ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2} + c$$

Now F(1) = 0

c = 0

$$y = -\ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2}$$

Curve lies in I^{st} quadrant so that option (1) & (2) are correct.

Q.7 Define the collection $\{E_1, E_2, E_3, ...\}$ of ellipses and $\{R_1, R_2, R_3, ...\}$ of rectangles as follows

$E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1;$

R_1 : rectangle of largest area, with sides parallel to the axes, inscribed in E_1 ;

E_n : ellipse $\frac{x^2}{a_n^2} + \frac{y^2}{b_n^2} = 1$ of largest area inscribed in R_{n-1}, n > 1;

R_n : rectangle of largest area, with sides parallel to the axes, inscribed in E_n, n > 1

Then which of the following options is/are correct ?

(1) $\sum_{n=1}^{N}(\text{area of } R_n) < 24$, for each positive integer N

(2) The eccentricities of E_{18} and E_{19} are NOT equal

(3) The length of latus rectum of E_9 is $\frac{1}{6}$

(4) The distance of a focus from the centre in E_9 is $\frac{\sqrt{5}}{32}$

Ans. **[1, 3]**

Sol. Area maximum when θ = 45º

	a	b
E_1	3	2
E_2	$\frac{3}{\sqrt{2}}$	$\frac{2}{\sqrt{2}}$
E_3	$\frac{3}{(\sqrt{2})^2}$	$\frac{2}{(\sqrt{2})^2}$

⋮

⋮

(1) $E_1 + E_2 + ... + E_m$

when $m \to \infty$ $\frac{2ab}{1-\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}} = 4ab = 4.3.2 = 24$

(2) Length of L.R. = $\frac{2b^2}{a} = \frac{2.4.24}{2.8.3} = \frac{1}{6}$

(3) Distance between focus and centre of ellipse of $I_9 = \frac{3}{2^4} . \frac{\sqrt{5}}{3} = \frac{\sqrt{5}}{16}$

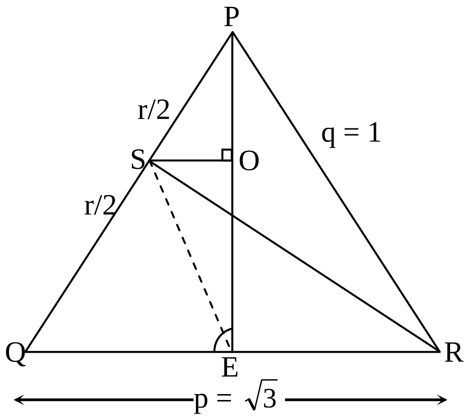
Q.8 In a non-right-angled triangle ΔPQR, let p, q, r denote the lenghts of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E and RS and PE intersect at O. If p = $\sqrt{3}$, q = 1, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct ?

(1) Radius of incircle of ΔPQR = $\frac{\sqrt{3}}{2}(2 - \sqrt{3})$ (2) Length of OE = $\frac{1}{6}$

(3) Area of ΔSOE = $\frac{\sqrt{3}}{12}$ (4) Length of RS = $\frac{\sqrt{7}}{2}$

Ans. **[1,2,4]**

Sol.



$\frac{p}{\sin P} = \frac{q}{\sin Q} = 2(1)$

$\Rightarrow \sin P = \frac{\sqrt{3}}{2}, \sin Q = \frac{1}{2}$

$\angle P = 60^o$ or 120^o

and $\angle Q = 30^o$ or 150^o

$\because \angle P + \angle Q < 180^o$ and $\neq 90^o$

$\angle P = 120^o$ & $\angle Q = 30^o$ & $\angle R = 30^o$

$\frac{r}{\sin R} = 2 \Rightarrow r = 1$

Medians RS = $\frac{1}{2}\sqrt{2p^2 + 2q^2 - r^2} = \frac{1}{2}\sqrt{6+2-1} = \frac{\sqrt{7}}{2}$

inradius $= \frac{2\Delta}{p+q+r} = \frac{\frac{2pqr}{4\times1}}{p+q+r} = \frac{1}{2}\left(\frac{1\times1\times\sqrt{3}}{1+1+\sqrt{3}}\right) = \frac{\sqrt{3}}{2}\left(\frac{2-\sqrt{3}}{1}\right)$

$\Rightarrow \frac{1}{2} \times \sqrt{3} \times PE = \frac{pqr}{4(1)}$ (equal area of Δ)

$$PE = \frac{1\times1\times\sqrt{3}}{4}\times\frac{2}{\sqrt{3}} = \frac{1}{2}$$

$$\Rightarrow OE = \frac{2(\text{area of } \Delta OPQ)}{QR} = \frac{2\times\frac{1}{3}\left(\frac{1}{2}\times1.\sqrt{3}\sin 30^\circ\right)}{\sqrt{3}} = \frac{1}{6}$$

SECTION – 3 (Maximum Marks : 18)

- This section contains **SIX (06)** questions. The answer to each question is a **NUMERICAL VALUE**.
- For each question, enter the correct numerical value of the answer using the mouse and the on-screen virtual numeric keypad in the place designated to enter the answer. If the numerical value has more than two decimal places, **truncate/round-off** the value to TWO decimal places.
- Answer to each question will be evaluated according to the following marking scheme.

  Full Marks : +3 If ONLY the correct numerical value is entered.

  Zero Marks : 0 In all other cases.

Q.1 Three lines are given by $\vec{r} = \lambda\hat{i}$, $\lambda \in R$; $\vec{r} = \mu(\hat{i}+\hat{j})$, $\mu \in R$ and $\vec{r} = \nu(\hat{i}+\hat{j}+\hat{k})$, $\nu \in R$. Let the lines cut the plane x + y + z = 1 at the points A, B and C respectively. If the area of the triangle ABC is Δ, then the value of $(6\Delta)^2$ equals ____

Ans. **[0.75]**

Sol. A(λ , 0, 0)

B(μ, μ, 0)

C(ν, ν, ν)

A, B & C satisfies x + y + z = 1

$\therefore \lambda = 1$

$\mu = \frac{1}{2}$

$\nu = \frac{1}{3}$

$\therefore$ A(1, 0, 0), B $\left(\frac{1}{2},\frac{1}{2},0\right)$, C$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$

$$\overrightarrow{AB} = -\frac{1}{2}\hat{i}+\frac{1}{2}\hat{j}+0\hat{k}$$

$$\overrightarrow{AC} = -\frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}+\frac{1}{3}\hat{k}$$

$$\Delta = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}| \Rightarrow \frac{1}{2}\left\|\begin{matrix}\hat{i} & \hat{j} & \hat{k}\\ -1/2 & 1/2 & 0\\ -2/3 & 1/3 & 1/3\end{matrix}\right\|$$

$\Delta = \frac{\sqrt{3}}{2\times 6}$

$\therefore\ 36\Delta^2 = \frac{3}{4} = 0.75$

Q.2 If $I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4}\frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$, then $27I^2$ equals ___

Ans. **[4.00]**

Sol. $I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4}\frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$

$2I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4}\frac{dx}{(2-\cos 2x)}$

$I = \frac{2}{\pi}\int_{0}^{\pi/4}\frac{dx}{3-2\cos^2 x}$

$I = \frac{2}{\pi}\int_{0}^{\pi/4}\frac{\sec^2 x}{1+3\tan^2 x}\,dx$

$\sqrt{3}\tan x = t$

$\sec^2 x\,dx = \frac{dt}{\sqrt{3}}$

$I = \frac{2}{\pi}\times\frac{1}{\sqrt{3}}\int_{0}^{\sqrt{3}}\frac{dt}{1+t^2}$

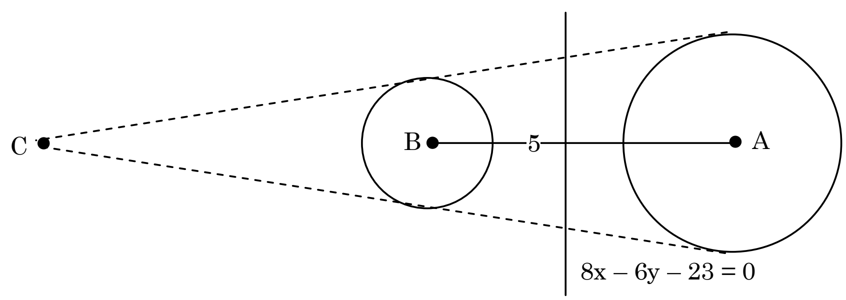
$I = \frac{2}{\sqrt{3}\pi}\times\frac{\pi}{3} = \frac{2}{3\sqrt{3}}$

$27I^2 = 27\times\frac{4}{9\times 3} = 4$

Q.3 Let the point B be the reflection of the point A(2, 3) with respect to the line $8x - 6y - 23 = 0$. Let Γ_A and Γ_B be circles of radii 2 and 1 with centres A and B respectively. Let T be a common tangent to the circles Γ_A and Γ_B such that both the circles are on the same side of T. If C is the point of intersection of T and the line passing through A and B, then the length of the line segment AC is ____.

Ans. **[10.00]**

Sol.



$AL = \left|\frac{16-18-23}{10}\right| = \frac{5}{2}$

$\frac{CB}{CA} = \frac{1}{2}$

$\frac{CA-5}{CA} = \frac{1}{2}$

CA = 10

Q.4 Let $\omega \neq 1$ be a cube root of unity. Then the minimum of the set $\{|a + b\omega + c\omega^2|^2 : a, b, c \text{ distinct non-zero integers}\}$ equals ___

Ans. **[3.00]**

Sol. $|a + b\omega + c\omega^2|^2 = a^2 + b^2 + c^2 - ab - bc - ca$

$$= \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

It will be minimum when a, b, c are consecutive integers.

So, minimum value is '3'

Q.5 Let AP(a; d) denote the set of all the terms of an infinite arithmetic progression with first term a and common difference d > 0. If AP(1 ; 3) ∩ AP(2; 5) ∩ AP(3 ; 7) = AP(a; d) then a + d equals ____

Ans. **[157.00]**

Sol. First series {1, 4, 7, 10, 13 .... } → c.d = 3

Second series {2, 7, 12, 17 .... } → c.d = 5

Third series {3, 10, 17, 24 .... } → c.d = 7

First common term of these series is 52

New common difference of new series is

L.C.M of (3, 5, 7) = 105

∴ a = 52

d = 105

a + d = 157

Q.6 Let S be the sample space of all 3 × 3 matrices with entries from the set {0, 1}. Let the events E_1 and E_2 be given by $E_1 = \{A \in S : \det A = 0\}$ and $E_2 = \{A \in S : \text{sum of entries of A is 7}\}$. If a matrix is chosen at random from S, then the conditional probability $P(E_1|E_2)$ equals ____

Ans. **[0.50]**

Sol. In matrix A there are 7 ones and 2 zeros

∴ Sum of elements of A = 7

Number of such matrices = $^9C_2$

Out of all such matrices E_1 will be those when both zeros lie in the same row or in the same column

eq. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$n(E_1 \cap E_2) = 2 \times {}^3C_2 \times {}^3C_2 = 18$

So, $n\left(\frac{E_1}{E_2}\right) = \frac{n(E_1 \cap E_2)}{n(E_2)} = \frac{18}{16} = \frac{1}{2} = 0.50$